Infinite Series Convergence Tests

Disclaimer: there is no general method of deciding what tests to use on a particular series; the sequence described here will usually help for the sorts of series we encounter in this course.

An absolute series \( \sum_{n=1}^{\infty} a_n \) is summed over positive terms \( a_n \). An alternating series \( \sum_{n=1}^{\infty} (-1)^n b_n \) has terms of the form \( a_n = (-1)^n \cdot b_n \), with the \( b_n \) factors being positive.

**Test for Divergence** -- if the limit of the positive term \( \lim_{n \to \infty} a_n \) or \( \lim_{n \to \infty} b_n \) is not zero or does not exist, then the series \( \sum_{n=1}^{\infty} a_n \) is divergent.

**Comparison Tests** (for series with positive terms) --

particularly effective for terms which are rational functions of polynomials (where the Ratio Test is not helpful)

(simple) **Comparison Test** -- if the general term of a series can be shown to be

smaller than the general term of a known convergent series, then the series is convergent; or

larger than the general term of a known divergent series, then the series is divergent

**Limit Comparison Test** -- if the general terms of a series of unknown convergence behavior and of a series where such behavior is known are put into a ratio, and it is found that \( \lim_{n \to \infty} \left( \frac{c_n}{d_n} \right) \) is a positive constant (either general term may be placed in the numerator or denominator), then the two series have the same convergence behavior (either both converge or both diverge)

**Ratio Test** --

this tests for absolute convergence

particularly effective for terms which contain exponential factors \( (a^n, n^n) \) or factors of \( n! \)

if the limit of the absolute value of the ratio of the next term in a series to the current term, \( \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| \),
exists and is less than 1, the series is absolutely convergent; exists and is greater than 1, the series is divergent; or exists and equals 1, the Ratio Test cannot resolve the question

**Alternating Series Test** --

an alternating series which fails to be absolutely convergent, but passes this test is said to be conditionally convergent

the conditions are that:

1 -- if $\lim_{n \to \infty} b_n = 0$ (this is equivalent to passing the Test for Divergence)

and 2 -- if $b_n \geq b_{n+1}$ (as demonstrated by direct verification of the inequality or finding that $\frac{d}{dn}(b_n) < 0$),

then the alternating series is convergent

**Root Test** --

this tests for absolute convergence

it is rather specialized, in that it only really works on general terms where all the factors are raised to the same power $n$; thus, we don’t make use of it very often

if the limit of the $n$th root of the absolute value of the general term, $\lim_{n \to \infty} \sqrt[n]{|a_n|}$, exists and is less than 1, the series is absolutely convergent; exists and is greater than 1, the series is divergent; or exists and equals 1, the Root Test cannot resolve the question

**Integral Test** --

this tests for absolute convergence

we don’t find so much use for this test in this course; we generally only try it when nothing else seems to help

it is applicable when the terms of the series are associated with a continuous, positive, and monotonically decreasing function, $f(n) = a_n$, beyond some finite value of $n$

the series $\sum_{n=N}^{\infty} a_n$ is linked to the Type I improper integral $\int_{N}^{\infty} f(x) \, dx$ and so has the same convergence behavior; thus, either the integral and the series both converge or both diverge