Math 1031
Solutions for Fall 2004 Final Exam

1) The first condition to consider in finding the domain of a rational function is that the function is undefined if the denominator is zero. For our expression, this means we cannot permit \( x - 1 = 0 \Rightarrow x = 1 \).

In addition, however, this function includes a square root, which will not be defined where the expression under the radical is less than zero. Thus, we also cannot allow \( 3 - x^2 < 0 \Rightarrow 3 < x^2 \Rightarrow \sqrt{3} < |x| \) (since \( \sqrt{x^2} = |x| \)).

The domain of our expression then does not include \( x = 1 \), \( x < -\sqrt{3} \), or \( x < \sqrt{3} \). The interval representing the domain is therefore \( (-\sqrt{3}, 1) \cup (1, \sqrt{3}] \). (A)

2) There are two ways we could solve this problem. We want the equation for a line parallel to \( 3x + 2y - 6 = 0 \) which passes through the point \((-1, -1)\). Two lines which are parallel have the same slope, so we must use the slope of the given line as the slope of our line.

One approach would be to re-arrange the standard form for the given line into slope-intercept form, which would tell us the slope directly:

\[
3x + 2y - 6 = 0 \Rightarrow 2y = -3x + 6 \Rightarrow y = -\frac{3}{2}x + 3.
\]

So the slope of the given line and our line is \( m = -\frac{3}{2} \). We can now use the point-slope form for the equation of a line to write

\[
(y - y_0) = m \cdot (x - x_0) \Rightarrow (y - [-1]) = -\frac{3}{2} \cdot (x - [-1])
\]

\[
\Rightarrow y + 1 = -\frac{3}{2} \cdot (x + 1) \Rightarrow y + 1 = -\frac{3}{2}x - \frac{3}{2} \Rightarrow y = -\frac{3}{2}x - \frac{5}{2}.
\]

This doesn’t match any of the choices available, so we will re-arrange this result into standard form to find

\[
y = -\frac{3}{2}x - \frac{5}{2} \Rightarrow 2y = -3x - 5 \Rightarrow 3x + 2y + 5 = 0.
\]

Another method is to use the fact that two lines with the same slope will have equations in standard form of \( ax + by + c = 0 \) in which only the value of \( c \) differs. Since the given line is \( 3x + 2y - 6 = 0 \), our line will be \( 3x + 2y + c = 0 \), for which we must find \( c \). Our line passes through \((-1, -1)\), so we must have \( 3 \cdot (-1) + 2 \cdot (-1) + c = 0 \)

\[
\Rightarrow (-3) + (-2) + c = 0 \Rightarrow c = 5.
\]

So the equation for our line is \( 3x + 2y + 5 = 0 \). (D)
3) At the end of the course, if the student wishes to have an average score of 90 on the four exams, the average would be calculated from \( \frac{87 + 94 + 80 + x}{4} = 90 \), with \( x \) being the score they receive on the last exam. We can then solve for the score they will need on the fourth exam, using
\[
87 + 94 + 80 + x = 4 \cdot 90 = 360 \\
\Rightarrow x = 360 - 87 - 94 - 80 = 99. 
\]

4) When the absolute value of an expression is involved in an equation, we must consider two cases: one in which the expression is positive or zero, and the other in which it is negative. In the first case, the absolute value signs can be removed, since \( |x - 5| = x - 5 \) for \( x - 5 \geq 0 \) (or \( x \geq 5 \)):
\[
x - 5 = x^2 - 1 \Rightarrow x^2 - x + 4 = 0 .
\]
But if we use the quadratic formula, we find that the discriminant is
\[
b^2 - 4ac = (-1)^2 - 4 \cdot 1 \cdot 4 = 1 - 16 = -15 < 0,
\]
so this equation has no real solutions. For the other case, however, the expression is negative, meaning \( -(x - 5) = -(x - 5) \) for \( x - 5 < 0 \) (or \( x < 5 \)), which gives us
\[
-x + 5 = x^2 - 1 \Rightarrow x^2 + x - 6 = 0 \\
\Rightarrow (x + 3) \cdot (x - 2) = 0 \Rightarrow x = -3 \text{ or } x = 2.
\]
These last two are then the complete solutions for our equation. \((A)\)

5) If the graph of a polynomial function has \( x \)-intercepts, we can find them by setting the function equal to zero and solving for the values of \( x \). For our function, we obtain
\[
y = x^2 + 3x - 10 = 0 \Rightarrow (x + 5) \cdot (x - 2) = 0 \Rightarrow x = -5, x = 2.
\]
The \( x \)-intercepts for our function are \((-5, 0)\) and \((2, 0)\), so the distance between them is \(7\). \((B)\)

6) As we did in Problem 4, we must look at the two cases produced by the presence of the absolute value term. In one case, \( |x - 1| = x - 1 \) for \( x - 1 \geq 0 \) (or \( x \geq 1 \)), making the inequality \( x - 1 \leq x \Rightarrow -1 \leq 0 \). Since this is always true, this means that the inequality is correct under the condition for which it applies; thus, \( x \geq 1 \). For the other case, \( |x - 1| = -(x - 1) \) for \( x - 1 < 0 \) (or \( x < 1 \)), the inequality becomes \( -(x - 1) \leq x \Rightarrow 1 - x \leq x \Rightarrow 1 \leq 2x \Rightarrow \frac{1}{2} \leq x \). Here, however, this result contradicts the condition for which the inequality was arrived at, so there is no consistent solution for this case. The original inequality is thus correct only for \( x \geq 1 \). \((C)\)
7) We could just try the various values of \( x \) offered in the choices to see which ones work in the polynomial, but it will be more efficient to solve the given equation directly. When all of the powers of \( x \) are even, it is possible to make the substitution \( t = x^2 \), giving us

\[
x^4 - x^2 - 2 = 0 \Rightarrow (x^2)^2 - x^2 - 2 = 0 \Rightarrow t^2 - t - 2 = 0
\]

\[
\Rightarrow (t + 1) \cdot (t - 2) = 0 \Rightarrow t = -1, t = 2
\]

The result of this substitution tells us that either \( x^2 = -1 \) or \( x^2 = 2 \). But the first of these equations has no solutions involving real numbers, while the second equation does have real solutions. So the solutions for our original equation which are real numbers are \( x = -\sqrt{2} \) and \( x = \sqrt{2} \). (D)

8) The distance between two points \((x_1, y_1)\) and \((x_2, y_2)\) in the plane is given by

\[
d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.
\]

Hence, the distance between the two given points is

\[
d = \sqrt{(3 - [-2])^2 + (11 - [-1])^2} = \sqrt{5^2 + 12^2} = \sqrt{25 + 144} = \sqrt{169} = 13.
\]

(B)

9) The circle in question has its equation expressed in standard form as

\[
x^2 + y^2 + 4x - 6y - 23 = 0.
\]

In order to be able to find easily the center and radius of the circle, we need to convert this equation by “completing the squares” in the expression, after first rearranging its terms:

\[
x^2 + 4x + y^2 - 6y = 23 \Rightarrow (x^2 + 4x + 4) + (y^2 - 6y + 9) = 23 + 4 + 9
\]

\[
\Rightarrow (x + 2)^2 + (y - 3)^2 = 36 = 6^2.
\]

We find from this re-writing of the equation of the circle that its center lies at \((-2, 3)\) and that its radius is 6. (E)

10) The meaning of \((g \circ f)(2)\) is that we should first evaluate the function \( f(x) \) at \( x = 2 \) to find \( f(2) \), which we’ll call \( X \) for the moment, and then evaluate the function \( g(x) \) at \( x = X \). In other words, \((g \circ f)(2) = g(X) = g(f(2))\).

We now need to investigate the results of the two functions involved. Since \( f(x) = 2x - 5 \), then we just have \( f(2) = (2 \cdot 2) - 5 = -1 \). The function \( g(x) \) has a branched definition, so the expression to use depends on the value of \( x \). For \( x < 1 \), \( g(x) = 2x + 2 \), so \((g \circ f)(2) = g(f(2)) = g(-1) = 2 \cdot (-1) + 2 = 0\). (B)
11) A direct way to find the inverse of a function \( y = f(x) \) is to “exchange the rôles” of \( x \) and \( y \) in the expression for the function and solve the result for \( y \) in terms of \( x \) (when possible). For our function, \( y = f(x) = \frac{1}{x+1} \), we would write

\[
y = \frac{1}{x+1} \quad \Rightarrow \quad x = \frac{1}{y+1} \quad \Rightarrow \quad y + 1 = \frac{1}{x} \quad \Rightarrow \quad y = \frac{1}{x} - 1 = f^{-1}(x) . \quad \text{(A)}
\]

We can check to see if this function works as the inverse of \( f(x) \) by evaluating

\[
(f^{-1} \circ f)(x) = f^{-1}(f(x)) = \frac{1}{f(x)} - 1 = \frac{1}{\frac{1}{x+1}} - 1 = (x+1) - 1 = x ,
\]

so we have found the correct inverse function (we would also find that \( f(f^{-1}(x)) = x \)).

12) There are two characteristics of the graph shown for the function \( f(x) = ax^3 + bx^2 + x \) which can provide us with information about the coefficients of the polynomial. The first is that the \( \text{large-}\lvert x\rvert \) behavior of the graph is to give large positive values of \( f(x) \) for large negative values of \( x \) and large negative values of the function for large positive values of \( x \). This is just the opposite of what happens for \( y = x^3 \); therefore, it must be the case that the coefficient \( a \) is negative. The second relevant feature of the graph is that the one zero of the function appears to be located at \( x = 0 \) and the other two are placed symmetrically about the \( y \)-axis. Thus, the function \( f(x) \) is symmetrical, which requires that the powers of \( x \) in the polynomial be either all even or all odd. Since \( a \neq 0 \), the powers must in fact all be odd, which requires that \( b = 0 \). This leaves choice \( \text{(D)} \).

13) We are told that the population of this town changed by a constant number of people every year, so we need to find this number in order to project into the past or future from the current population. If the population was 15,200 at the beginning of 1991 and 15,890 exactly three years later at the start of 1994, then this constant rate of growth is given by

\[
\]

We can now use this information to find the population at the start of 1989; we can work from

\[
r = \frac{P(1994) - P(1989)}{1994 - 1989} = \frac{15,890 - P(1989)}{5} = 230
\]

\[
\Rightarrow \quad 15,890 - P(1989) = 5 \cdot 230 = 1150 \quad \Rightarrow \quad P(1989) = 15,890 - 1150 = 14,740 \quad . \quad \text{(C)}
\]

We can also find this final result working backward from the 1991 population over an interval of 2 years.
14) The way we solve this problem in this course is to exploit the properties of the parabola. The equation \( h = 24t - 16t^2 \) describes a “downward-facing” parabola. The vertex, which will mark the maximum height for the thrown ball, lies on the symmetry axis of the parabola, halfway between the \( t \)-intercepts. We can find these intercepts by setting the height equal to zero:

\[
h = 24t - 16t^2 = 0 \quad \Rightarrow \quad t \cdot (24 - 16t) = 0 \quad \Rightarrow \quad t = 0 \quad \text{or} \quad \frac{24}{16} = \frac{3}{2}.
\]

The symmetric axis thus lies at \( t = \frac{0 + \frac{3}{2}}{2} = \frac{3}{4} \) second, so the vertex is found at \( \left( \frac{3}{4}, h\left(\frac{3}{4}\right) \right) = \left( \frac{3}{4}, 24 \cdot \frac{3}{4} - 16 \cdot \left[ \frac{3}{4} \right]^2 \right) = \left( \frac{3}{4}, 18 - 16 \cdot \frac{9}{16} \right) = \left( \frac{3}{4}, 9 \right) \). Thus, the ball reaches a maximum height of 9 feet.

15) We approach this problem by examining how we might re-write the number \( 7 \sqrt{35} \) in terms of the numbers 5 and 7. Using the fact that \( \sqrt{ab} = \sqrt{a} \cdot \sqrt{b} \), we have \( 7 \sqrt{35} = 7 \cdot (\sqrt{7}) \cdot (\sqrt{5}) \). If we now apply logarithms to this equation, this becomes

\[
\log_a 7 \sqrt{35} = \log_a [7 \cdot \sqrt{7} \cdot \sqrt{5}] = \log_a 7 + \log_a \sqrt{7} + \log_a \sqrt{5} = \log_a 7 + \log_a 7^{1/2} + \log_a 5^{1/2} = \log_a 7 + \frac{1}{2} \log_a 7 + \frac{1}{2} \log_a 5
\]

\[
= \frac{3}{2} \log_a 7 + \frac{1}{2} \log_a 5.
\]

If we designate \( b = \log_a 5 \) and \( c = \log_a 7 \), then we can write \( \log_a 7 \sqrt{35} = \frac{b}{2} + \frac{3c}{2} \).

16) For the sample described in the problem, the amount of mass that is still carbon-14 is given by \( m(t) = 10 \cdot e^{-t/8223} \), with \( m \) being the mass of \(^{14}\text{C} \) in grams and \( t \) being the time that has passed, in years. If we call \( T \) the number of years at which only one gram of carbon-14 remains in the sample, then we can solve for this time using

\[
m(T) = 1 = 10 \cdot e^{-T/8223} \quad \Rightarrow \quad e^{-T/8223} = \frac{1}{10} \quad \Rightarrow \quad -T/8223 = \ln \frac{1}{10}
\]

\[
\Rightarrow \quad T = -8223 \ln \frac{1}{10} = 18,934 \text{ years}.
\]
17) In order to solve this equation, it will be helpful to simplify it first using the properties of logarithms:

\[ \log_2[(x + 1)^{5/6}] - \log_2[(x + 1)^{1/2}] = 2 \implies \frac{5}{6} \log_2(x + 1) - \frac{1}{2} \log_2(x + 1) = 2 \implies \frac{5}{6} - \frac{1}{2} \log_2(x + 1) = 2 \implies \frac{1}{3} \log_2(x + 1) = 2 \implies \log_2(x + 1) = 6 \]

exponentiate both sides \[ a^{\log a^u} = u \]

\[ 2^{\log_2(x + 1)} = 2^6 \implies x + 1 = 64 \implies x = 63 \quad \text{(E)} \]

18) We will first consider the number of ways in which we can produce a bridge hand containing 13 cards, in which 11 of the cards are from a single one of the suits and the remaining two from a second suit. The eleven cards would be drawn from among the thirteen of that suit, so there would be 13 choices for the first card drawn, 12 for the second one, and so on down to 3 for the last of those eleven. However, the order of the cards is of no importance, since our only concern is about which cards were drawn. So there are \( 13 \cdot 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \) ways in which the eleven cards could have been drawn, but there are \( 11! \) possible orders in which the set of eleven could appear, all of which we count as identical. The number of distinct sets of eleven cards taken from the suit are then \( \frac{13 \cdot 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3}{11!} = \frac{13!}{11!} \) or \( \frac{13!}{11!2!} \), which is just \( \frac{13 \cdot 12}{2} \). In a similar fashion, we can consider the two cards drawn from the second suit of thirteen cards: there are \( 13 \cdot 12 \) ways in which these two cards could have been drawn and \( 2! \) distinct sequences in which the two could have appeared, making the number of distinct set of two cards from that suit equal to \( \frac{13 \cdot 12}{2} = \frac{13!}{2!11!} = \frac{13!}{2!11!} \), which again gives us \( \frac{13 \cdot 12}{2} \).

We have as yet said nothing about the two suits themselves. The first suit, say, the one from which we took eleven cards, could be one of the four suits available, while the second suit would be one of the three suits left, from which we drew two cards. Here, the order of the suits does matter, since taking eleven hearts and two clubs is different from getting eleven clubs and two hearts. Thus, the total number of bridge hands which we have described in this problem is

\[ 4 \cdot 3 \cdot \frac{13 \cdot 12}{2} \cdot \frac{13 \cdot 12}{2} = 3 \cdot 13^2 \cdot 12^2 \quad (= 73,008) \quad \text{(E)} \]

Considering that the number of possible bridge hands is \( \frac{52!}{13!39!} = 6.35 \cdot 10^{11} \), the odds of drawing such a hand is about one chance in 8.7 million.

19) The probability of occurrence of a particular type of event is given by the ratio of the number of outcomes of interest to the number of all possible outcomes. We know that the number of possible outcomes for the rolling of two dice is \( 6 \cdot 6 = 36 \). We also need to find the number of outcomes for which the numbers shown by the two dice differ by at least 4. Since the dice can only show the integers one through six, the sole
outcomes of interest here are for the dice to show 6 and 2, 6 and 1, or 5 and 1. For each of these results, there are two possible orderings, since the two dice can show 6 and 2 or 2 and 6, for example. Altogether then, there are 6 relevant outcomes out of the total possible 36, so the probability for the type of event described is $1/6$. (A)

20) The most direct way to determine the probability of failing the test by not stopping at any one of the six stop signs is to find first the probability of successfully stopping at all of them. Since the success or failure of stopping at any one sign is assumed to be independent of (that is, has no effect on) the success or failure at any other sign, and the probability of stopping successfully is $1 - 0.1 = 0.9$ each time, the probability of stopping properly at all six signs is $(0.9)^6 \approx 0.531$. Therefore, the probability of failing to stop at any of the six signs is $1 - 0.531 \approx 0.469$. (B)

21) A conditional probability gives the chance that an event A will occur, provided that an event B has occurred (the events may or may not be related). This conditional probability is found from the equation

$$P(A \mid B) = \frac{P(A \text{ and } B)}{P(B)}.$$ 

We are told that there are 10,000 deer being tested for a disease. The number which are found to “test positive” is 15 in all, so the probability that a randomly selected deer among this group is one of these is $P(\text{test positive}) = 15/10000$. It is found in fact, presumably through further medical study, that only 3 deer are actually infected, but just two of those tested positive. So the probability that one of the randomly selected deer out of the entire group is both infected and tested positive is $P(\text{infected and tested positive}) = 2/10000$. The probability then that a deer chosen at random from this population is infected, given that it tests positive for the disease is given by

$$P(\text{infected} \mid \text{tests positive}) = \frac{P(\text{infected})}{P(\text{tests positive})} = \frac{2/10000}{15/10000} = \frac{2}{15}. \quad (D)$$

Another approach to finding this probability is to note that, altogether, 15 deer tested positive, but only two of that group are infected. So the probability that a deer which tests positive is infected is $2/15$. (The medical test being described in this problem would be considered rather poor for the purpose of screening for the disease.)

22) The expected value for a game is the average of winnings and losses, weighted according to their probabilities; if there are $n$ different possible events, each of which pays an amount $w_i$ ($w < 0$ represents a loss) and has a probability $p_i$ of occurring, then the expected value of the game is

$$E = w_1 \cdot p_1 + w_2 \cdot p_2 + \cdots + w_n \cdot p_n = \sum_{i=1}^{n} w_i \cdot p_i.$$ 

A game is considered “fair” if the expected value is $E = 0$; a game which has $E < 0$ “favors the house” (such games keep casinos, fair concessions and lotteries in business).

In the rules of this game, we toss two fair coins (the likelihoods of a coin showing heads or tails are equal). The probability of both coins showing heads is then $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$, as is the probability that both show tails; there are two ways in which the coins can show one head and one tail, so the probability of this happening is $2 \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{2}$.

(continued)
We pay two dollars to play one round of the game. If both coins come up heads, then our winning is the net gain $5 - 2 = 3$. When the coins show one head and one tail, we have a net loss of $1 - 2 = -1$. Finally, if both coins show tails, we get no money back and our net loss is $0 - 2 = -2$.

The expected value of this game is then

$$E = 3 \cdot \frac{1}{4} + (-1) \cdot \frac{1}{2} + (-2) \cdot \frac{1}{4} = 0.75 + (-0.5) + (-0.5) = (-0.25).$$

On the average, we lose 25 cents per game, so the game slightly favors the party running it.

23) The cleanest approach to solving an equation of this sort is to isolate the term with the radical first, and then square both sides in order to remove the square root. We should also be aware that the presence of a square root restricts the possible solutions to the equation, since square roots of negative numbers do not give real numbers. It is required then that $31 - 9x \geq 0 \Rightarrow 31 \geq 9x \Rightarrow \frac{31}{9} \geq x$; we must keep this in mind in regarding our solutions below.

We arrange our equation to leave the radical term alone on one side:

$$x + \sqrt{31 - 9x} = 5 \Rightarrow \sqrt{31 - 9x} = 5 - x.$$ 

If we now square both sides of the equation, we obtain

$$\left(\sqrt{31 - 9x}\right)^2 = (5 - x)^2 \Rightarrow 31 - 9x = 25 - 10x + x^2 \Rightarrow x^2 - x - 6 = 0 \Rightarrow (x - 3)(x + 2) = 0 \Rightarrow x = -2 \text{ or } x = 3.$$ 

Since both numbers are less than $\frac{31}{9}$, both solutions are valid for our equation.

**check:**

- $-2 + \sqrt{31 - 9 \cdot (-2)} = -2 + \sqrt{31 + 18} = -2 + \sqrt{49} = -2 + 7 = 5$ ;
- $3 + \sqrt{31 - 9 \cdot 3} = 3 + \sqrt{31 - 27} = 3 + \sqrt{4} = 3 + 2 = 5$ . ✓

24) In solving inequalities, it is generally best to arrange for all the terms to be on just one side of the inequality, so that the result can be compared to zero; this allows us to make some conclusions based on the sign of the collected terms. For our inequality, we would have

$$\frac{2}{x} \geq \frac{1}{x + 3} \Rightarrow \frac{2}{x} - \frac{1}{x + 3} \geq 0 \Rightarrow \frac{2(x + 3) - x}{x(x + 3)} \geq 0$$

$$\Rightarrow \frac{2x + 6 - x}{x(x + 3)} \geq 0 \Rightarrow \frac{x + 6}{x(x + 3)} \geq 0.$$ 

(continued)
The most direct way to solve such an inequality is to find first the “special points” in the function on the left-hand side. The ratio equals zero when \( x = -6 \), which makes the numerator zero, so this solves the equation portion of the inequality; this also marks a boundary for the intervals we need to consider. The ratio is undefined when the denominator is zero, which occurs when \( x = 0 \) or \( x = -3 \); these values mark the two other boundaries for intervals.

We can now look at the signs of the various terms in the ratio for the intervals \( x < -6 \), \( -6 < x < -3 \), \( -3 < x < 0 \), and \( x > 0 \):

\( x < -6 \) -- for large negative numbers, the factors \( x + 6 \), \( x + 3 \), and \( x \) are all negative; so we have a negative number in the numerator and a product of two negative numbers in the denominator; thus, the numerator is negative and the denominator is positive, so the ratio is negative in this interval \( \left( \frac{\_}{\_-\_-} = \_ = \_ \right) \); hence, the interval does not solve the inequality;

\( -6 < x < -3 \) -- here, the term \( x + 6 \) is positive, so the ratio is now positive \( \left( \frac{\_+}{\_-\_} = \_ = \_ = \_ \right) \); this interval, then, is a solution to the inequality;

\( -3 < x < 0 \) -- now, \( x + 6 \) is still positive, but \( x + 3 \) has also become positive, making the product in the denominator negative, and thus the ratio becomes negative again \( \left( \frac{\_+}{\_-\_} = \_ = \_ = \_ \right) \); this means that this interval is not in the solution for the inequality;

\( x > 0 \) -- finally, for positive values of \( x \), all of the factors in the ratio are positive, hence the entire ratio is positive \( \left( \frac{\_+}{\_+} = \_ = \_ \right) \); therefore, this interval does solve the inequality.

When we put these results together with the fact that \( x = -6 \) solves the equation portion, we find that the intervals for the solution of the inequality are \( (-6, -3) \) and \( (0, \infty) \).

We can instead use specific “test numbers” in each of the intervals in order to check whether the inequality is satisfied or not, but this “signs method” requires much less computation, since it is only the signs of the factors and of the ratio that we are interested in.

25) We can analyze the situation in this way. The radiator is filled with 6 quarts of coolant, one-fifth of which is anti-freeze and the rest of which is water. We need to remove some amount \( x \) of this solution, in order to make room for the pure anti-freeze we want to add. When we're done, we intend the radiator to be filled with a half-and-half solution, which means there will then be 3 quarts of anti-freeze and 3 quarts of water.

When we have drained off \( x \) quarts of solution from the radiator, there will remain \( 6 - x \) quarts, of which one-fifth is anti-freeze, or \( \frac{1}{5}(6 - x) \) quarts. We will then add \( x \) quarts to refill the radiator to capacity, all of this being pure anti-freeze. Once
we have done this, the radiator will hold 6 quarts of coolant again, with 3 quarts of this now being anti-freeze. This requires then that
\[
\frac{1}{3}(6-x) + x = 3 \implies \frac{6}{3} - \frac{1}{3}x + x = 3 \implies \frac{4}{3}x = \frac{9}{3} \implies x = \frac{9}{4} \text{ or } 2.25 \text{ quarts .}
\]
check: removing 2.25 quarts of coolant from the radiator leaves 6 - 2.25 = 3.75 quarts, of which \( \frac{1}{3} \cdot 3.75 = 0.75 \) quarts is anti-freeze; adding 2.25 quarts of pure anti-freeze restores the volume of coolant in the radiator to 6 quarts, of which 0.75 + 2.25 = 3 quarts is now anti-freeze, making a 50-50 solution in the radiator.

26) In applying the given exponential growth model, \( N(t) = 1000 \cdot e^{kt} \), we need to evaluate the “growth constant” \( k \). If we work with \( t \) as time in months from the start of 1997, then we know that at the beginning of the year (\( t = 0 \)), there were
\[
N(0) = 1000 \cdot e^{k \cdot 0} = 1000 \cdot e^0 = 1000 \cdot 1 = 1000 \text{ cases ,}
\]
and that ten months later, the number of cases had doubled, or
\[
N(10) = 1000 \cdot e^{k \cdot 10} = 2000 \text{ cases .}
\]
If we now take a ratio of this two results, we find
\[
\frac{N(10)}{N(0)} = \frac{1000 \cdot e^{10k}}{1000} = \frac{2000}{1000} \implies e^{10k} = 2
\]
\[
\implies 10k = \ln 2 \implies k = \frac{1}{10} \ln 2 \approx 0.0693 .
\]
taking logarithms of both sides
After twelve months (the end of 1997), the number of cases is thus predicted to reach
\[
N(12) = 1000 \cdot e^{12k} = 1000 \cdot e^{\left( \frac{12}{10} \ln 2 \right)} = 1000 \cdot e^{\left( \frac{12}{10} \ln 2 \right)}
\]
\[
= 1000 \cdot e^{\left( \ln 2 \right) \cdot \frac{6}{5}} = 1000 \cdot \left( e^{\ln 2} \right)^{6/5} = 1000 \cdot 2^{6/5} \approx 2297 \text{ cases .}
\]
or \( 1000 \cdot e^{12 \cdot (0.0693)} = 1000 \cdot e^{0.8316} \approx 1000 \cdot 2.297 \)

27) a) If we start from the polynomial \( y = -3x^2 + 6x + 6 \), we can extract information about the location of the parabola's vertex by “completing the square”:
\[
y = -3x^2 + 6x + 6 = -3 \cdot (x^2 - 2x) + 6 = -3 \cdot (x^2 - 2x + 1) + 6 - [-3 \cdot 1]
\]
\[
= -3 \cdot (x - 1)^2 + 6 - (-3) = -3 \cdot (x - 1)^2 + 9 \implies y - 9 = -3 \cdot (x - 1)^2 .
\]
This result tells us that we have a “downward-facing” parabola with its vertex at (1, 9).

(continued)
We can use this form of the equation for the parabola to find the $x$-intercepts of its graph by setting $y = 0$ to obtain

$$0 - 9 = -3 \cdot (x - 1)^2 \Rightarrow \frac{-9}{-3} = 3 = (x - 1)^2 \Rightarrow x - 1 = \pm \sqrt{3} \Rightarrow x = 1 \pm \sqrt{3}.$$  

(This is equivalent to solving the quadratic formula.) We can return to the original polynomial to find the $y$-intercept by setting $x = 0$; hence, $y = -3 \cdot 0^2 + 6 \cdot 0 + 6 = 6$ (which is to say that the constant term of the polynomial gives us the $y$-intercept).

So the parabola “faces downward”, has its vertex at $(1, 9)$, its $x$-intercepts at $(1 \pm \sqrt{3}, 0)$, and its $y$-intercept at $(0, 6)$. A graph is presented in the Answer Key.

b) The range of a function is the set of all numbers that the function can possibly produce. Since our parabola “faces downward” from a vertex of $(1, 9)$, the polynomial produces no value for $y$ that is larger than 9. So the range of our function is $y \leq 9$.

c) From the form of our parabola, we see that the function increases for values of $x$ less than 1 and decreases for values of $x$ greater than 1. (At $x = 1$, $f(x)$ is considered to be neither increasing nor decreasing.)

28)  
a) We are told that among the seeds which are sown in this year (2004), twice as many are those saved from 2002 as those saved from 2003. So the seeds from 2002 make up $2/3$ of all the seeds that are being sown. The probability that a seed that is selected at random from those being sown came from the flowers of 2002 is thus $2/3$.

b) Here, we need to consider the viability of the seeds saved from different years. If we choose at random a seed from those to be sown, there is a $2/3$ chance that it is from 2002 and $1/3$ that it is from 2003. If the seed we’ve selected is from 2002, it has only a 30% probability of germinating successfully; otherwise, the seed is from 2003 and has a 90% probability of germination. The probability that the seed we’ve taken will germinate, then, is found by combining these factors, gives us

$$P(\text{will germinate}) = P(2002) \cdot P(2002 \ and \ will \ germinate) + P(2003) \cdot P(2003 \ and \ will \ germinate)$$

$$= \frac{2}{3} \cdot 0.3 + \frac{1}{3} \cdot 0.9 = 0.2 + 0.3 = 0.5 \ or \ \frac{1}{2}.$$