Math 1031
Solutions for Spring 2005 Final Exam

1) The probability that a particular type of event occurs is the ratio of the number of outcomes involving that type of event to the number of all possible outcomes. For the “flipping” of a coin, the number of possible sequences of results is \(2^6 = 64\). To find the probability of getting at least two heads, we must find the number of outcomes in which two, three, four, five, or six heads show.

In situations such as this, however, it may not be very convenient to count up all of these possibilities. It can sometimes be easier instead to count all the outcomes which do not meet the desired conditions and then to subtract those from all the possible outcomes; the result will be the number of outcomes which do satisfy our condition. We could then just consider the number of ways in which we could get zero heads (TTTTTT – just one way) or one head (HTTTTT or the one coin showing heads at any of the five other positions – six ways). Thus, there are 7 outcomes in which less than two heads show.

We can then express the probability we seek as

\[
P(\text{six coins, two or more heads}) = \frac{\text{number of outcomes with two or more heads}}{\text{total number of outcomes}} \]

\[= \frac{(\text{total number of outcomes}) - (\text{number of outcomes with less than two heads})}{\text{total number of outcomes}} \]

\[= \frac{64 - 7}{64} = \frac{57}{64}. \]

This can also be expressed as

\[P(\text{six coins, two or more heads}) = 1 - P(\text{six coins, less than two heads}) = 1 - \frac{7}{64} = \frac{57}{64}. \]

2) The clearest method for determining the line perpendicular to the given line is first to express the equation for that line in slope-intercept form:

\[9x + 2y = 1 \quad \Rightarrow \quad 2y = -9x + 1 \quad \Rightarrow \quad y = -\frac{9}{2}x + \frac{1}{2}. \]

The slope of this line is thus \(m = -\frac{9}{2}\). Two lines are perpendicular to one another is the product of their slopes is \(m \cdot m' = -1\); the slope for our line is then given by \(m' = -\frac{1}{m} = -\frac{1}{-\frac{9}{2}} = \frac{2}{9}\). (This information already eliminates all of the choices except for (D) and (E)).

We desire our line to pass through the point \((-4, 2)\), so we can use the slope we’ve found in the point-slope form equation

\[y - y_0 = m'(x - x_0) \quad \Rightarrow \quad y - 2 = \frac{2}{9} \cdot (x - [-4]) \]

\[\Rightarrow \quad y - 2 = \frac{2}{9}x + \frac{8}{9} \quad \Rightarrow \quad y = \frac{2}{9}x + \frac{8}{9} + \frac{18}{9} = \frac{2}{9}x + \frac{26}{9}. \quad (D)\]
3) A commonly-used approach for finding the inverse of a function (when it is algebraically possible) is to “exchange the rôles” of $x$ and $y$ in the expression of the function, and then to solve the resulting expression for $y$:

\[
y = \frac{x}{2x-1} \quad \Rightarrow \quad x = \frac{y}{2y-1}
\]

\[
\Rightarrow x \cdot (2y-1) = y \quad \Rightarrow \quad 2xy - x = y \quad \Rightarrow \quad 2xy - y = x
\]

\[
\Rightarrow (2x-1) \cdot y = x \quad \Rightarrow \quad y = \frac{x}{2x-1} = f^{-1}(x) . \quad (B)
\]

Indeed, we find that $f(x) = f^{-1}(x)$, that is, this function is its own inverse! (Such functions are called self-inverse.) We can check this by composing the functions

\[
f^{-1} \circ f(x) = f^{-1}(f(x)) = \left(\frac{x}{2x-1}\right) = 2 \cdot \left(\frac{x}{2x-1}\right) - 1
\]

\[
= \left(\frac{2x}{2x-1}\right) = \left(\frac{1}{2x-1}\right) = x ,
\]

which also equals $f \circ f^{-1}(x)$ and $f \circ f(x)$.

4) The formula for calculating compound interest when compounded “continuously” is $A = P \cdot e^{rt}$, where $A$ is the current amount, $P$ is the principal, $r$ is the annual interest rate expressed as a fraction, and $t$ is the time in years. We want to calculate the amount of time in years, $T$, at which $\$600$ compounded continuously at 8% annual interest attains the value of $\$2400$. The equation is set up as

\[
2400 = 600 \cdot e^{0.08 \cdot T} \quad \Rightarrow \quad e^{0.08 \cdot T} = \frac{2400}{600} = 4 \quad \Rightarrow \quad 0.08 \cdot T = \ln 4 \quad \Rightarrow \quad T = \frac{\ln 4}{0.08} . \quad (A)
\]

taking logarithms of both sides

5) This problem just requires the straightforward use of algebra. Multiplication of the equation by each of the denominators, or cross-multiplication of the ratios, is safe, since any solutions this equation may have will not involve either of the denominators becoming zero:

\[
\frac{1}{2y+3} = \frac{4}{5y+6} \quad \Rightarrow \quad 5y + 6 = 4 \cdot (2y+3) \quad \Rightarrow \quad 5y + 6 = 8y + 12
\]

\[
\Rightarrow \quad 3y = -6 \quad \Rightarrow \quad y = -2 . \quad (E)
\]

So there is a solution to the equation, but it is not listed among the choices.
6) An equation or inequality which uses the absolute value operation actually involves two cases: one in which the expression inside the absolute value brackets is positive or zero, and the other in which the expression is negative. We must solve each of these cases separately:

\[ 1 - 5x \geq 0 \rightarrow \mathrm{In\ this\ case,\ } 1 < 1 - 5x \Rightarrow 5x < 0 \Rightarrow x < 0 \ ; \ or \]
\[ 1 - 5x < 0 \rightarrow \mathrm{Here,\ } 1 < -(1 - 5x) \Rightarrow 1 < 5x - 1 \Rightarrow 5x > 2 \Rightarrow x > \frac{2}{5} . \]

The two of these inequalities together constitute the solution of our inequality. (D)

7) The domain of a function is the set of (real) numbers for which it is possible to calculate a value of the function. Since \( f(x) \) here is a rational function, it is defined anywhere that the denominator is not zero. Since the denominator is an odd root, \( \sqrt[3]{x + 1} \), it is defined for all real numbers, and so presents no restriction on the domain of our function. The numerator, however, is an even root, \( \sqrt{5 - x^2} \), which is only defined where the argument of the radical is positive or zero; this is the case for
\[ 5 - x^2 \geq 0 \Rightarrow 5 \geq x^2 \Rightarrow \sqrt{5} \geq |x| \ \mathrm{or} \ -\sqrt{5} \leq x \leq \sqrt{5} . \]

This interval, then, marks the domain of \( f(x) \), which can also be written as \( [-\sqrt{5} , \sqrt{5}] \). (A)

8) We will apply the properties of logarithms to the expression \( \ln \left( (12x)^2 \right) \) to see if we can make it resemble one of the choices:

\[ \log_a u^p = p \log_a u \quad \text{factoring argument} \quad \log_a (uv) = \log_a u + \log_a v \]
\[ \ln \left( (12x)^2 \right) = 2 \cdot \ln (12x) = 2 \cdot [\ln (2^2 \cdot 3 \cdot x)] = 2 \cdot [\ln (2^2) + \ln (3) + \ln x] \]
\[ = 2 \cdot [2 \ln 2 + \ln 3 + \ln x] = 4 \ln 2 + 2 \ln 3 + 2 \ln x = 4 \ln 2 + 2 \ln (3x) . \] (B)

9) Since this polynomial equation contains only even powers of \( x \), it is possible to make a substitution \( t = x^2 \), in order to transform it into a quadratic equation:
\[ x^4 + x^2 - 12 = 0 \rightarrow t^2 + t - 12 = 0 \Rightarrow (t + 4) \cdot (t - 3) = 0 \]
\[ \Rightarrow t = -4 , \ t = 3 \ \rightarrow \ x^2 = -4 , \ x^2 = 3 . \]

The first of these resulting equations has two solutions which are imaginary numbers \( (2i \ \text{and} \ -2i) \), and thus cannot be counted toward the number of real solutions. The second equation has the two real solutions, \( -\sqrt{3} \ \text{and} \ \sqrt{3} \), so these are the only real solutions to the original equation. (A)
The expected value for a game is the average of winnings and losses, weighted according to their probabilities; if there are \( n \) different possible events, each of which pays an amount \( w_i (w_i < 0 \) represents a loss) and has a probability \( p_i \) of occurring, then the expected value of the game is

\[
E = w_1 \cdot p_1 + w_2 \cdot p_2 + \ldots + w_n \cdot p_n = \sum_{i=1}^{n} w_i \cdot p_i .
\]

A game is considered “fair” if the expected value is \( E = 0 \); a game which has \( E < 0 \) “favors the house” (such games keep casinos, fair concessions and lotteries in business).

For this game, we will need to consider the different possible events, their likelihoods, and the gains or losses associated with them. Since the outcomes do not depend upon the suit of the card drawn, we can assess the probabilities simply in terms of how many of each type of card appears in a single suit:

<table>
<thead>
<tr>
<th>rank of card</th>
<th>probability</th>
<th>gain or loss</th>
</tr>
</thead>
<tbody>
<tr>
<td>ace</td>
<td>1/13</td>
<td>-$25</td>
</tr>
<tr>
<td>face card (king, queen, jack)</td>
<td>3/13</td>
<td>-$10</td>
</tr>
<tr>
<td>10</td>
<td>1/13</td>
<td>+$10</td>
</tr>
<tr>
<td>9</td>
<td>“</td>
<td>+$9</td>
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<tr>
<td>8</td>
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<tr>
<td>5</td>
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<td>4</td>
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<tr>
<td>3</td>
<td>“</td>
<td>+$3</td>
</tr>
<tr>
<td>2</td>
<td>“</td>
<td>+$2</td>
</tr>
</tbody>
</table>

The expected value of the game then works out to be

\[
E = \frac{1}{13} (-$25) + \frac{3}{13} (-$10) + \frac{1}{13} ($10) + \frac{1}{13} ($9) + \ldots + \frac{1}{13} ($2) \\
= \frac{1}{13} (-$25 + $10 + $9 + $8 + $7 + $6 + $5 + $4 + $3 + $2) + \frac{3}{13} (-$10) \\
= -$\frac{29}{13} - $\frac{30}{13} = -$\frac{1}{13} .
\]

Thus, this game slightly favors the party offering it.

11) We are asked to calculate the result of the composition of two functions, \( f \circ g (x) = f (g(x)) \), so we must make this evaluation in stages. For the composition \( f \circ g (-1) \), we first need to find \( g(-1) \): this is simply

\[
g(-1) = (-1)^3 - 4 \cdot (-1) = (-1) + 4 = 3 .
\]

Since the function \( f(x) \) has a branched definition, the expression we must use to evaluate it depends on the value of \( x \). For \( x > 0 \), \( f(x) = 2 - 2x \), so we find that

\[
f \circ g (-1) = f (g(-1)) = f (3) = 2 - 2 \cdot 3 = -4 .
\]
It can be difficult to extract some kinds of information about a conic section (ellipse/circle, parabola, or hyperbola) from the equation for the curve when it is given in standard form. So we will need to use “completion of squares” to re-write the equation in a more helpful form:

\[ x^2 + y^2 - 8x + 2y = 32 \quad \Rightarrow \quad x^2 - 8x + y^2 + 2y = 32 \]

\[ \Rightarrow (x^2 - 8x + 16) + (y^2 + 2y + 1) = 32 + 16 + 1 \]

\[ \Rightarrow (x - 4)^2 + (y + 1)^2 = 49 = 7^2. \]

We can now tell immediately from this form of the equation for the circle that its center lies at (4, -1) and that it has a radius of 7.

The event described here is the result of a sequence of independent outcomes due to rolling a single fair die three times and flipping a single fair coin five times. The order of the outcomes matters here, since the event is described by a specific set of consequences; thus, we do not need to do any counting of arrangements. Because all of the outcomes are independent of one another, we can simply take the probability of a particular outcome for rolling the die or tossing the coin and multiply them all together. Therefore, the probability of the sequence described is

\[ P(1, 2, 3, H, T, T, T, H) = P(1) \cdot P(2) \cdot P(3) \cdot P(H) \cdot P(T) \cdot P(T) \cdot P(T) \cdot P(H) \]

\[ = \frac{1}{6} \cdot \frac{1}{6} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{6^3 \cdot 2^5}. \]

The number of textbooks (5) is smaller than the number of people (7) among whom they are to be distributed. There is no restriction placed on how many books any of the individuals may receive. Therefore the transfer of each book can be made to any of the seven people each of five times, and each transfer is independent of any of the others. So the number of ways the books may be distributed is \( 7 \cdot 7 \cdot 7 \cdot 7 = 7^5 \).

A conditional probability gives the chance that an event A occurs, provided that an event B has occurred (the two events may or may not be related); this is computed using the formula \( P(A|B) = \frac{P(A \text{ and } B)}{P(B)} \). For this problem, we are interested in finding the probability that the tossing of four fair coins shows four heads, given that we already know that at least two of the coins show heads. Through the formula, we would express this as

\[ P(\text{four heads} \mid \text{at least two heads}) = \frac{P(\text{four heads and at least two heads})}{P(\text{at least two heads})}. \]

We need to consider the outcomes of tossing four coins in order to evaluate the probabilities required in the formula. There are \( 2^4 = 16 \) possible outcomes from tossing the coins. Of these, there is just one outcome in which four heads are obtained, so \( P(\text{four heads}) = \frac{1}{16} \); because getting four heads means we did get at least two tells us that \( P(\text{four heads and at least two heads}) = P(\text{four heads}) \). We still need to find the
chance of getting at least two heads from tossing four coins. We could count up the number of cases in which two, three, or four heads result, but it is less work simply to eliminate the cases in which zero or two heads are obtained:

**zero heads** -- this is equivalent to getting four tails, which can only occur in one way, as getting four heads does;

**one head** -- the one head can occur on any one of the four coins, so there are four ways that this type of event can occur

There are then five outcomes out of the total of 16 for which we get less than two heads, so there are eleven outcomes in which we end up with at least two heads. Hence we obtain the probability \( P(\text{at least two heads}) = \frac{11}{16} \). We are now able to evaluate the probability through the conditional probability formula to find

\[
P(\text{four heads} | \text{at least two heads}) = \frac{1/16}{11/16} = \frac{1}{11}.
\]

We can also find this result by noting that, of the eleven outcomes for which the four coins produce at least two heads, only one of these outcomes gives us four heads. Hence, the conditional probability sought is \( 1/11 \).

16) a) We can calculate these probabilities by considering what needs to occur on each draw from the deck and the (independent) probability of each of those events. To get exactly two aces in four cards drawn, we can start by considering one possible situation. We draw one of the four aces in the deck on the first try; the probability of doing this is \( 4/52 \). Suppose we now draw one of the three remaining aces on the second try; since there are 51 cards left, the probability of this happening is \( 3/51 \). For the next draw, there are now 50 cards, 48 of which are not aces; since we only want a hand with exactly two aces, we want to draw any of the other cards this time, the chance of which is \( 48/50 \). Finally, on the fourth draw, we again want anything other than an ace, which represents 47 out of the 49 cards left; the probability of not getting an ace this time is then \( 47/49 \). So the chance of having the sequence of cards \( \text{AAXX} \) (‘X’ being any card which is not an ace) is

\[
\frac{4}{52} \cdot \frac{3}{51} \cdot \frac{48}{50} \cdot \frac{47}{49}.
\]

This is not the only sequence of four cards containing exactly two aces, of course; we must still count all of the possible arrangements of the four cards. Consider the set of four as having four places where one of the aces could have appeared. Once that position is taken, there remain three places for the second ace, so the number of arrangements of the four cards would be \( 4 \cdot 3 \). However, the order in which the aces were drawn doesn’t matter, since we are only concerned that the hand have exactly two and it is not important whether the ace of clubs came up in the second draw and the aces of hearts on the fourth, or the other way around. So the number of sequences of the two aces among the four cards is

\[
\frac{4 \cdot 3}{2} = \frac{4!}{2!2!} = \binom{4}{2}.
\]

(Another way of thinking about this is that we ignore the ordering of the aces in the counting of sequences since we already allowed for the different orders when we counted four possible aces for one draw and three for the second.) The probability for the hand is then

\[
\text{(continued)}
\]
\[ P(\text{exactly two aces}) = \frac{4!}{2!2!} \cdot \frac{4}{52} \cdot \frac{3}{51} \cdot \frac{48}{50} \cdot \frac{47}{49} = \frac{6768}{270725} \approx 0.0250 . \]

b) The calculation for the probability of drawing a hand with no aces in it is similar to that in part (a). Now we want to draw cards other than aces each of the four times, so we could have one of 48 cards out of the 52 in the deck on the first draw, 47 out of 51 on the second, 46 out of 50 for the third draw, and 45 out of 49 on the last one (in other words, the four aces are left aside as cards not permissible to draw each time). The sequence would now be described as \( \text{XXXX} \); since the ordering of the cards which are not aces is unimportant, this is counted as the only sequence. Hence, the probability of drawing a four-card hand with no aces is

\[ P(\text{no aces}) = \frac{48}{52} \cdot \frac{47}{51} \cdot \frac{46}{50} \cdot \frac{45}{49} = \frac{38916}{54145} \approx 0.7187 . \]

c) We can carry out this calculation similarly to the ones above. Since we now want all four cards to be diamonds, which make up 13 of the 52 cards in the deck, we would need to draw one of these on the first draw, one of the remaining 12 diamonds out of the 51 cards left in the second draw, 11 out of 50 for the next draw, and 10 out of 49 for the last. The sequence for our hand is \( \text{DDDD} \), for which the order is not important, so it is counted as the only sequence (alternatively, we can say that we have already counted the order for the diamonds in the hand when we allowed one of 13 the first time, one of 12 on the second draw, and so on). We find the probability of drawing an all-diamond four-card hand is thus

\[ P(\text{four diamonds}) = \frac{13}{52} \cdot \frac{12}{51} \cdot \frac{11}{50} \cdot \frac{10}{49} = \frac{11}{4165} \approx 0.00264 . \]

17) It is usually easiest to analyze inequalities by first arranging for all the terms to be on just one side of the inequality, so that the result can be compared to zero. We can then make conclusions based on the sign of the collected terms (which is much easier to think about than trying to find relationships among the given ratios). For our inequality, we would have

\[
\frac{3}{x-1} \geq \frac{1}{x+2} \quad \Rightarrow \quad \frac{3}{x-1} - \frac{1}{x+2} \geq 0 \quad \Rightarrow \quad \frac{3 \cdot (x+2) - 1 \cdot (x-1)}{(x-1) \cdot (x+2)} \geq 0
\]

\[
\Rightarrow \quad \frac{3x + 6 - x + 1}{(x-1) \cdot (x+2)} \geq 0 \quad \Rightarrow \quad \frac{2x + 7}{(x-1) \cdot (x+2)} \geq 0 .
\]

There are “special points” for the function on the left-hand side that we can now locate. The ratio equals zero when \( x = -\frac{7}{2} \), which makes the numerator zero, so this solves the equation portion of the inequality and marks a boundary for the intervals we need to consider. The ratio is undefined when the denominator is zero, which occurs when \( x = 1 \) or \( x = -2 \); these values mark the two other boundaries for intervals.
We can now look at the signs of the various terms in the ratio for the intervals \( x < -\frac{7}{2}, -\frac{7}{2} < x < -2, -2 < x < 1, \) and \( x > 1 \):

- \( x < -\frac{7}{2} \) -- for large negative numbers, the factors \( 2x + 7, x + 2, \) and \( x - 1 \) are all negative; so we have a negative number in the numerator and a product of two negative numbers in the denominator; thus, the numerator is negative and the denominator is positive, so the ratio is negative in this interval \( \left( -\frac{7}{2}, -1 \right) \); hence, the interval does not solve the inequality;

- \( -\frac{7}{2} < x < -2 \) -- here, the term \( 2x + 7 \) is positive, so the ratio is now positive \( \left( \frac{+}{-} = \frac{+}{-} = + \right) \); this interval, then, is a solution to the inequality;

- \( -2 < x < 1 \) -- now, \( 2x + 7 \) is still positive, but \( x + 2 \) has also become positive, making the product in the denominator negative, and thus the ratio becomes negative again \( \left( \frac{+}{-} = \frac{+}{-} = - \right) \); this means that this interval is not in the solution for the inequality;

- \( x > 1 \) -- finally, for positive values of \( x - 1 \), all of the factors in the ratio are positive, hence the entire ratio is positive \( \left( \frac{+}{+} = \frac{+}{+} = + \right) \); therefore, this interval does solve the inequality.

When we put these results together with the fact that \( x = -\frac{7}{2} \) solves the equation portion, we find that the intervals for the solution of the inequality are \( \left( -\frac{7}{2}, -2 \right) \) and \( (1, \infty) \).

We can instead use specific “test numbers” in each of the intervals in order to check whether the inequality is satisfied or not, but this “signs method” requires much less computation, since it is only the signs of the factors and of the ratio that we are interested in.

18)

a) An equation containing a radical is best dealt with by isolating the radical on one side of the equation before raising it to the appropriate power in order to eliminate the root operation. From that point on, algebra will handle the rest:

\[
x + \sqrt{33 - 8x} = 3 \quad \Rightarrow \quad \sqrt{33 - 8x} = 3 - x \quad \Rightarrow \quad (\sqrt{33 - 8x})^2 = (3 - x)^2
\]

\[
\Rightarrow \quad 33 - 8x = 9 - 6x + x^2 \quad \Rightarrow \quad x^2 + 2x - 24 = 0 \quad \Rightarrow \quad (x + 6) \cdot (x - 4) = 0
\]

This last result suggests that the solutions to the equation are \( x = -6 \) and \( x = 4 \).

(continued)
However, we must consider other issues here. When solving an equation containing a square root by squaring both sides, there is the potential for “creating a solution” which does not work in the original equation we were to solve.

One concern is that the square root operation is only defined (for real numbers) when the argument of the radical is positive or zero. So a solution could be “created” which is not in the domain of the radical. We see that any solution to the original equation must satisfy $33 - 8x \geq 0 \Rightarrow 33 \geq 8x \Rightarrow \frac{33}{8} \geq x$. For this problem, both $-6$ and $4$ meet this requirement, so they are all right so far.

The other difficulty is that squaring both sides of the equation causes us to treat positive and negative square roots of a number on an equal footing thereafter, even when both roots may not work in the original equation. That is just what we find here upon testing both of our candidate solutions:

$x = -6: \quad (-6) + \sqrt{33 - 8 \cdot (-6)} = (-6) + \sqrt{33 + 48} = (-6) + \sqrt{81} = (-6) + 9 = 3 \; \checkmark$

$x = 4: \quad 4 + \sqrt{33 - 8 \cdot 4} = 4 + \sqrt{33 - 32} = 4 + \sqrt{1} = 4 + 1 = 5 \neq 3 \; \times$

Thus, $x = 4$ does not solve our original equation, making the lone solution $x = -6$.

b) To solve a logarithmic equation, it will generally be convenient to collect all of the logarithmic terms (of the same base) into a single term and then to perform an exponentiation of both sides of the equation to “invert” the logarithmic operation. The resulting equation can then be solved through algebraic methods.

By applying the properties of logarithms, we obtain

$$
\log_3(x - 8) + \log_3 x = 2 \Rightarrow \log_3[(x - 8) \cdot x] = 2 \Rightarrow (x - 8) \cdot x = 3^2
$$

$$
\begin{align*}
\log_a u + \log_a v &= \log_a (uv) \\
\text{exponentiate both sides} \\

x^2 - 8x &= 9 \Rightarrow x^2 - 8x - 9 = 0 \Rightarrow (x + 1) \cdot (x - 9) = 0 .
\end{align*}
$$

So it would appear that the solutions are $x = -1$ and $x = 9$. However, as with “squaring an equation” as we did in part (a), “exponentiating an equation” can generate false solutions which do not work in the original equation. Since the logarithmic operation is only defined for positive real numbers, the two logarithmic terms in our equation cannot be applied to the value $x = -1$. Consequently, the only solution to our equation is $x = 9$.

19) If we “complete the square” in the quadratic function for this parabola, it will become easier to answer some of the questions:

$$
y = 3x^2 + x - 2 = 3 \cdot (x^2 + \frac{1}{3}x) - 2 = 3 \cdot (x^2 + \frac{1}{3}x + \frac{1}{36}) - 2 - 3 \cdot \frac{1}{36}
$$

$$
y = 3 \left(x + \frac{1}{6}\right)^2 - 2 - \frac{1}{12} \Rightarrow y + \frac{25}{12} = 3 \left(x + \frac{1}{6}\right)^2 .
$$

(continued)
We can now read off immediately that the vertex of the parabola lies at \( (-\frac{1}{6}, -\frac{25}{12}) \) [part (b)]. The \( x \)-intercept(s), if they exist, can be found by setting \( y = 0 \) in our resulting equation and solving for \( x \) (this is equivalent to solving the original function using the quadratic formula):

\[
0 + \frac{25}{12} = 3(x + \frac{1}{6})^2 \Rightarrow (x + \frac{1}{6})^2 = \frac{1}{3} \cdot \frac{25}{12} = \frac{25}{36} \Rightarrow \sqrt{(x + \frac{1}{6})^2} = \pm \frac{5}{6}
\]

\[
\Rightarrow x + \frac{1}{6} = \pm \frac{5}{6} \Rightarrow x = -\frac{1}{6} \pm \frac{5}{6} \Rightarrow x = -1 \text{ or } x = \frac{2}{3}.
\]

The \( x \)-intercepts are found then at \((-1, 0)\) and \(\left(\frac{2}{3}, 0\right)\). We can use the original equation for the parabola to find its \( y \)-intercept by setting \( x = 0 \), yielding \( y = 3 \cdot 0^2 + 0 - 2 = -2 \) (or we could just note the constant term of the polynomial), which tells us that this intercept is at \((0, -2)\) [part (a)].

\[\textbf{c)}\text{ Since the coefficient of the quadratic term for the parabola’s function is positive, we see that this parabola is “upward-facing”. Together with the other information we have obtained, we can sketch a graph of the parabola, as shown in the Answer Key.}\]

\[\textbf{20)}\]

\[\textbf{a)}\text{ We are told that 35\% of all the recordable CDs have come from supplier N and that of these 2\% are found to be defective. This implies that 2\% of 35\% of all the CD-Rs which this computer store receives are the ones that are both from supplier N and are defective. This is a fraction } 0.02 \cdot 0.35 = 0.007 \text{ of all the CD-Rs at the store, and thus represents the probability that a randomly selected CD-R from this store is one of those.}\]

\[\textbf{b)}\text{ The proportion of CD-Rs that are acceptable (not defective) can be found by combining the proportions of such discs which have been obtained from each supplier:}\]

\[
\text{fraction of acceptable discs} = (\text{fraction from supplier M}) \cdot (\text{acceptable fraction from M}) + (\text{fraction from supplier N}) \cdot (\text{acceptable fraction from N})
\]

\[
= (0.65) \cdot (1 - 0.09) + (0.35) \cdot (1 - 0.02)
\]

\[
= (0.65) \cdot (0.91) + (0.35) \cdot (0.98)
\]

\[
= 0.5915 + 0.343 = 0.9345
\]

This is the fraction of discs in the store that are in acceptable condition, and thus also the probability that a CD-R randomly selected at this store is good.
21) The function for the height of the ball above the ground has a graph which is a "downward-facing" parabola. We can find its t-intercepts (since the graph will be of height \( h \) as a function of time \( t \)) by setting \( h = 0 \) and solving for the values of \( t \):

\[
h = 0 = -16t^2 + 40t = t \cdot (-16t + 40) \implies t = 0 \text{ or } -16t + 40 = 0 \implies t = \frac{40}{16} = \frac{5}{2} = 2.5.
\]

So the ball is at ground level at \( t = 0 \) (it is common in physics problems to treat the ball as if it were thrown starting from right on the ground) and again at \( t = 2.5 \) seconds [part(b)].

Since the symmetry axis of the parabola is midway between its t-intercepts, the vertex of this parabola is found at \( t = \frac{0 + \frac{5}{2}}{2} = \frac{5}{4} \) or 1.25 seconds. The maximum height of the ball is then given by

\[
h\left(\frac{5}{4}\right) = -16 \cdot \left(\frac{5}{4}\right)^2 + 40 \cdot \left(\frac{5}{4}\right) = -16 \cdot \frac{25}{16} + 40 \cdot \frac{5}{4} = -25 + 50 = 25 \text{ feet} \] [part(a)].

G. Ruffa – 8/09