1) This is just a straightforward differentiation problem. Since the derivative of a sum of functions is the sum of the derivatives of the functions, we can differentiate the functions making up \( f(x) \) "term-by-term":

\[
f'(x) = \frac{d}{dx} (x^3 + e^x + \ln x + 5) = \frac{d}{dx} (x^3) + \frac{d}{dx} (e^x) + \frac{d}{dx} (\ln x) + \frac{d}{dx} (5)
\]

\[
= 3x^2 + e^x + \frac{1}{x} + 0 = 3x^2 + e^x + \frac{1}{x}. \quad \text{(B)}
\]

2) The domain of a rational function \( R(x) = \frac{f(x)}{h(x)} \) consists of the values of \( x \) which are in both the domain of \( f(x) \) and the domain of \( h(x) \), less those values which produce \( h(x) = 0 \). For our function, \( g(x) = \frac{x - 5}{\sqrt{x + 2}} \), the function in the numerator, \( f(x) = x - 5 \), is always defined, so its domain is all real numbers \( (\mathbb{R}) \), while the function in the denominator \( h(x) = \sqrt{x + 2} \) is only defined for \( x + 2 \geq 0 \Rightarrow x \geq -2 \). But further, the ratio that gives us \( g(x) \) will not be defined if \( x + 2 = 0 \), so we cannot permit \( x = -2 \). The domain of \( g(x) \) is then \( x > -2 \). \quad \text{(D)}

3) We can use either of two methods we have learned for determining this “limit at infinity”. For a rational function of polynomials, direct application of the Limit Laws will give us the indeterminate ratio \( \frac{\infty}{\infty} \), so instead we use the technique of dividing the numerator and denominator functions by the largest power of \( x \) appearing in the denominator and applying the special limit law \( \lim_{x \to \infty} \frac{1}{x^p} = 0 \). For our function, the largest power of \( x \) is \( x^2 \), so we have

\[
\lim_{x \to \infty} \frac{-3x^4 - 5}{10x^2 + 2x + 3} = \lim_{x \to \infty} \frac{-3x^4 - 5}{10x^2 + 2x + 3} \cdot \frac{1/x^2}{1/x^2} = \lim_{x \to \infty} \frac{-3x^4 - 5}{10x^2 + 2x + 3} \cdot \frac{1}{x^2} = \lim_{x \to \infty} \frac{-3x^4}{10x^2 + 2x + 3} \cdot \frac{x^2}{x^2}.
\]

\[
= \lim_{x \to \infty} \frac{-3x^2 - \frac{5}{x^2}}{10 + \frac{2}{x} + \frac{3}{x^2}} = \frac{\left( \lim_{x \to \infty} -3x^2 \right) - 0}{10 + 0 + 0} = \lim_{x \to \infty} -\frac{3}{10} x^2.
\]

Now, the function \( x^2 \) goes to “positive infinity” whether \( x \) approaches either positive or negative infinity. Since it is multiplied by the negative coefficient \( -\frac{3}{10} \), however, the limit is “negative infinity” for our rational function. \quad \text{(A)}
We can also resolve the indeterminate ratio for our limit by applying L'Hôpital's Rule:

\[
\lim_{x \to \infty} \frac{-3x^4 - 5}{10x^2 + 2x + 3} = \lim_{x \to \infty} \frac{(-3x^4 - 5)'}{(10x^2 + 2x + 3)'} = \lim_{x \to \infty} \frac{-12x^3 - 5}{20x + 2} = \frac{-\infty}{\infty}.
\]

We find in this case, however, that this leads us to another indeterminate ratio. But that means that we can use the Rule again:

\[
\lim_{x \to \infty} \frac{-12x^3 - 5}{20x + 2} = \lim_{x \to \infty} \frac{(-12x^3 - 5)'}{(20x + 2)'} = \lim_{x \to \infty} \frac{-36x^2}{20}.
\]

By an argument similar to the one we used above, this function will run to \(-\infty\) as \(x\) goes to either \(+\infty\) or \(-\infty\).

4) The statement of this problem includes an expression which is the limit definition of the derivative of the function \(f(x)\) at \(x = 1\), \(f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h}\). So we do not have to carry out the limit calculation, but can simply evaluate the derivative function at \(x = 1\):

\[
f(x) = 3x^2 + \sqrt{x} = 3x^2 + x^{1/2} \quad \Rightarrow \quad f'(x) = 6x + \frac{1}{2}x^{-1/2}
\]

\[
\Rightarrow \quad f'(1) = 6 \cdot 1 + \frac{1}{2} \cdot 1^{-1/2} = 6 + \frac{1}{2} = \frac{13}{2}.
\]

(C)

5) Here, we have a limit for which applying the Limit Laws gives us the indeterminate ratio \(0/0\). While it would seem convenient to use L'Hôpital's Rule here, as we did in Problem 3, the square-root in the denominator of this rational function cannot be removed in a useful way from the calculation (radicals are an “Achilles' heel” of the Rule). So we will instead see what we can accomplish by factoring the expression:

\[
\lim_{x \to 4} \frac{x^2 - 4x}{\sqrt{x} - 2} = \lim_{x \to 4} \frac{x(x - 4)}{\sqrt{x} - 2} = \lim_{x \to 4} \frac{x(\sqrt{x} + 2)(\sqrt{x} - 2)}{\sqrt{x} - 2} = \lim_{x \to 4} \frac{x(\sqrt{x} + 2)(\sqrt{x} - 2)}{\sqrt{x} - 2} = \lim_{x \to 4} x(\sqrt{x} + 2) = 4(\sqrt{4} + 2) = 4 \cdot 4 = 16.
\]

(D)

6) Many kinds of curves are described by equations involving the variables \(x\) and \(y\) which cannot be re-arranged conveniently (or, sometimes, at all) into the form of an explicit function \(y = f(x)\). Since there is one (or more) functions which are nevertheless expressed by the equation implicitly, these functions do have derivative functions. These may be found, through application of the Chain Rule, in what is called “implicit differentiation”. We differentiate both sides of the given curve equation to obtain
\[
\frac{d}{dx}\left[ \frac{(2x + y)^3}{u} \right] = \frac{d}{dx}(x^2 + 7) \Rightarrow \frac{d}{du}(u^3) \cdot \frac{du}{dx} = \frac{d}{dx}(x^2 + 7)
\]

\[
\Rightarrow \quad 3u^2 \cdot \frac{d}{dx}(2x + y) = \frac{d}{dx}(x^2 + 7) \Rightarrow \quad 2 + \frac{dy}{dx} = \frac{2x}{3u^2}
\]

\[
\Rightarrow \quad \frac{dy}{dx} = \frac{2x}{3(2x + y)^2} - 2.
\]

This result tells us the slope at any point of the curve, provided we are given both coordinates \(x\) and \(y\) (while we could find an explicit function for this particular curve, the expression for \(\frac{dy}{dx}\) in terms of \(x\) alone would be rather unpleasant).

Since we are just evaluating the slope of the curve at the point \((1, 0)\), we don’t need to “tidy up” the algebra in our expression. We find the slope of the tangent line at the specified point to be

\[
\left. \frac{dy}{dx} \right|_{(1,0)} = \frac{2 \cdot 1}{3(2 \cdot 1 + 0)^2} - 2 = \frac{2}{3 \cdot 2^2} - 2 = \frac{2}{12} - 2
\]

\[
= \frac{1}{6} - 2 = -\frac{11}{6}.
\]

(E)

7) We are working here with a “piecewise-defined” function, which has a different definition for each of its applicable intervals. In the two intervals \((-\infty, 1]\) and \((1,\infty)\), the function is defined by a polynomial. Since polynomial functions are always defined, our function \(f(x)\) can only have a potential “complication” at \(x = 1\), namely, that it could be discontinuous there.

The requirement for continuity at a point \(x = a\) is that \(\lim_{x \to a} f(x) = f(a)\); for this to be possible, the function must be defined at \(x = a\), the two-sided limit \(\lim_{x \to a} f(x)\) must exist, and that limit must equal \(f(a)\). We have already noted that \(f(1)\) is defined; from the definition on the interval \((-\infty, 1]\), the value is \(f(1) = 2 \cdot 1^2 - 2 = 1\). This is also, then, the value of the one-sided limit “from below”, \(\lim_{x \to 1^-} f(x)\). However, the one-sided limit “from above” uses the definition from the “branch” of the function on \((1, \infty)\), so \(\lim_{x \to 1^+} f(x) = 3 \cdot 1 + 2 = 5\). From this, we see that \(\lim_{x \to 1^-} f(x) \neq \lim_{x \to 1^+} f(x)\); so the two-sided limit \(\lim_{x \to 1} f(x)\) does not exist, and therefore \(f(x)\) is discontinuous at \(x = 1\).

(B)

8) We have a “Type I" improper integral, which has an upper limit of integration that extends to “positive infinity”. Such definite integrals must be evaluated by taking a “limit at infinity” of the anti-derivative function. The integration itself can be made by using a "u-substitution", which leads to
\[
\int_1^{\infty} \frac{1}{(2x+1)^3} \, dx \rightarrow \int_3^\infty \frac{1}{u^3} \left( \frac{1}{2} \, du \right) = \frac{1}{2} \int_3^\infty u^{-3} \, du = \lim_{t \to \infty} \frac{1}{2} \left( \frac{u^{-2}}{-2} \right) \bigg|_3^t
\]
\[u = 2x + 1, \quad du = 2 \, dx\]

\[
= \left[ \lim_{t \to \infty} \left( -\frac{1}{4t^2} \right) \right] - \left( -\frac{1}{4 \cdot 3^2} \right) = 0 - \left( -\frac{1}{36} \right) = \frac{1}{36}. \quad (A)
\]

9) The average value of a function \( f(x) \) over an interval \([a, b]\) is equivalent to the height of a rectangle with its base extending from \( x = a \) to \( x = b \) and having the same signed area as the function has over that interval. This means that we divide the definite integral of \( f(x) \) over the interval by the width of the interval to obtain

\[
\{ f(x) \}_{[a, b]} = \frac{\int_a^b f(x) \, dx}{b-a}.
\]

For the given function \( f(x) = (\sqrt{x})(x+1) \), its average value on the interval \([0, 1]\) is given by

\[
\{ f(x) \}_{[0,1]} = \frac{\int_0^1 \sqrt{x} \cdot (x+1) \, dx}{1-0} = \frac{\int_0^1 x^{1/2} \cdot (x+1) \, dx}{1} = \int_0^1 x^{3/2} + x^{1/2} \, dx
\]

\[
= \left( \frac{x^{5/2}}{5/2} + \frac{x^{3/2}}{3/2} \right) \bigg|_0^1 = \left( \frac{2}{5} \cdot 1^{5/2} + \frac{2}{3} \cdot 1^{3/2} \right) - \left( \frac{2}{5} \cdot 0^{5/2} + \frac{2}{3} \cdot 0^{3/2} \right)
\]

\[
= \frac{2}{5} + \frac{2}{3} = \frac{6 + 10}{15} = \frac{16}{15}. \quad (E)
\]

10) In order to find the area between two curves, we need to find the intersection points between the two of them to establish the limits of integration, and which of the two is the "upper curve" in the integration interval so that the difference between the functions can be taken in the proper order. For the given functions (which represent parabolas), \( y = f(x) = x^2 - 4 \) and \( y = g(x) = -2 - x^2 \), the intersection points are found by setting the two functions equal to one another:

\[
x^2 - 4 = -2 - x^2 \Rightarrow 2x^2 = -2 - (-4) = 2 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1.
\]

The two curves meet at \( x = -1 \) and \( x = 1 \), so the interval for integration is \([-1, 1]\). The “upper curve” in this interval is represented by the function which has the largest values throughout the interval. This can be established by calculating with whatever value of \( x \) in the interval is most convenient to work with, say, \( x = 0 \):

\[
f(0) = 0^2 - 4 = -4 < g(0) = -2 - 0^2 = -2,
\]
so \( g(x) > f(x) \) everywhere in the interval \((-1, 1)\), making \( g(x) \) the “upper curve”. The area between these curves is thus given by

\[
\int_{-1}^{1} g(x) - f(x) \, dx = \int_{-1}^{1} (-2 - x^2) - (x^2 - 4) \, dx = \int_{-1}^{1} (2 - 2x^2) \, dx
\]

\[
= (2x - \frac{2x^3}{3}) \bigg|_{-1}^{1} = (2 \cdot 1 - \frac{2 \cdot 1^3}{3}) - (2[-1] - \frac{2 \cdot [-1]^3}{3})
\]

\[
= \frac{4}{3} - (-\frac{4}{3}) = \frac{8}{3}.
\]

(C)

11) Since \( f(x) \) is a product of two terms which are polynomials, the derivative function is calculated through the application of the Product Rule and the Chain Rule:

\[ f(x) = (x^2 + 1)^3 (2x^2 - 1) \]

\[
\frac{d}{dx} f(x) = \frac{d}{dx} \left(\frac{x^2 + 1}{u}\right)^3 (2x^2 - 1) + (x^2 + 1)^3 \cdot \frac{d}{dx} (2x^2 - 1)
\]

Product Rule

\[
= \frac{d}{du} (u^3) \cdot \frac{du}{dx} \cdot (2x^2 - 1) + (x^2 + 1)^3 \cdot 4x
\]

Chain Rule

\[
= 3u^2 \cdot \frac{d}{dx} (x^2 + 1) \cdot (2x^2 - 1) + (x^2 + 1)^3 \cdot 4x
\]

\[
= 3 (x^2 + 1)^2 \cdot (2x) \cdot (2x^2 - 1) + (x^2 + 1)^3 \cdot 4x
\]

We are only going to evaluate this derivative function at \( x = 1 \), so we do not need to “clean up” the algebra. We find

\[
f'(1) = [3 (x^2 + 1)^2 \cdot (2x) \cdot (2x^2 - 1) + (x^2 + 1)^3 \cdot 4x] \bigg|_{x=1}
\]

\[
= 3 (1^2 + 1)^2 \cdot (2 \cdot 1) \cdot (2 \cdot 1^2 - 1) + (1^2 + 1)^3 \cdot 4 \cdot 1
\]

\[
= 3 (2^2) \cdot 2 \cdot 1 + (2^3) \cdot 4 = 24 + 32 = 56.
\]

(D)

12) For the requested partial differentiation with respect to \( y \), the variable \( x \) in the expression for the function \( f(x, y) \) is treated as if it were a constant; the Quotient Rule, however, applies in the usual way. We thus have
\[ f_y = \frac{\partial}{\partial y} \left( \frac{2xy}{x + 3y} \right) = \frac{(x + 3y) \cdot \frac{\partial}{\partial y} (2xy) - 2xy \cdot \frac{\partial}{\partial y} (x + 3y)}{(x + 3y)^2} \]

\[ = \frac{(x + 3y) \cdot (2x) - 2xy \cdot (0 + 3)}{(x + 3y)^2} = \frac{2x^2 + 6xy - 6xy}{(x + 3y)^2} = \frac{2x^2}{(x + 3y)^2}. \]

(Ordinarily, it isn’t necessary to “tidy up the algebra” on the result for the derivative if we are only interested in evaluating it. In this case, though, it was reasonably easy to see that we could make a simplification.) At \( x = 1 \), \( y = 1 \), the value of this partial derivative is \( f_y \bigg|_{(1,1)} = \left[ \frac{2x^2}{(x + 3y)^2} \right]_{(1,1)} = \frac{2 \cdot 1^2}{(1 + 3 \cdot 1)^2} = \frac{2}{4^2} = \frac{1}{8}. \) (B)

13) For a function of one variable, \( y = f(x) \), we locate “critical points” by determining the values of \( x \) for which \( \frac{df}{dx} = 0 \). We extend this to a function of two variables, \( z = f(x, y) \), by finding the values of \( x \) and \( y \) which simultaneously solve the equations for \( \frac{\partial f}{\partial x} = 0 \) and \( \frac{\partial f}{\partial y} = 0 \) (since we are working with a polynomial, there are no critical points for which the derivatives are undefined). The equations arising from the given function are

\[ \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} (x^2 + 2y^2 - 4xy + 6x) = 2x + 0 - 4y + 6 = 0 \Rightarrow 2x - 4y = -6, \]

\[ \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} (x^2 + 2y^2 - 4xy + 6x) = 0 + 4y - 4x + 0 = 0 \Rightarrow 4x - 4y = 0. \]

This pair of equations is straightforward to solve: the second one tells us that \( 4x = 4y \Rightarrow x = y \), which, by inserting this into the first equation, implies \( 2x - 4x = -6 \Rightarrow -2x = -6 \Rightarrow x = 3 = y \). This function thus has only the single critical point at \( (3, 3) \).

Returning to functions of one variable, we can evaluate the concavity of the curve at critical points by inspecting the sign of the second derivative: if \( \frac{d^2f}{dx^2} > 0 \), the critical point is a local minimum; if \( \frac{d^2f}{dx^2} < 0 \), it is a local maximum. With functions of two variables, however, we must check the behavior of second derivatives along each “dimension”, leading to more complicated possibilities. We now evaluate an index value \( \mathcal{D} = f_{xx} \cdot f_{yy} - (f_{xy})^2 \) at the critical point, which variously informs us that is a local minimum for \( \mathcal{D} > 0 \) with \( f_{xx} > 0 \), a local maximum for \( \mathcal{D} > 0 \) with \( f_{xy} < 0 \), and a “saddle point” for \( \mathcal{D} < 0 \) (a maximum along one dimension, a minimum along the other). We find the second partial derivative functions to be
Applying the Quotient Rule, we can locate any local extrema by calculating

\[ f_{xx} = \frac{\partial}{\partial x} (2x - 4y + 6) = 2 \quad \text{and} \quad f_{xy} = \frac{\partial}{\partial y} (2x - 4y + 6) = -4 \quad , \]

and \( f_{yy} = \frac{\partial}{\partial y} (4y - 4x) = 4 \) .

The index is then \( D = (2)(4) - (-4)^2 = 8 - 16 = -8 < 0 \) , entirely independently of the values of the coordinates \((x,y)\) of a critical point. Consequently, the critical point \((3, 3)\) for our function is a \textbf{saddle point}. (C)

14) We are told that the population of the bacterial culture follows exponential growth, which tells us that the number of bacteria at time \((\text{in hours}) \ t \geq 0 \) is given by \( N(t) = N(0) \cdot e^{kt} \). It is given that the initial population is \( N(0) = 1000 \); we do not know the “exponential growth constant” \( k \), but we know that \( N(2) = 4000 \). This allow us to see up the equation \( 4000 = 1000 \cdot e^{2k} \Rightarrow e^{2k} = 4 \Rightarrow \ln(e^{2k}) = \ln 4 \Rightarrow 2k = \ln 4 \Rightarrow k = \frac{1}{2} \ln 4 = \ln(4^{1/2}) = \ln 2 \). After

\[ p \log_a x = \log_a x^p \]

three hours, the population of the culture will have reached \( N(3) = 1000 \cdot e^{(\ln 2)^3} \). Since \( 3 \ln 2 = \ln (2^3) = \ln 8 \), we find that \( N(3) = 1000 \cdot e^{(\ln 8)} = 1000 \cdot 8 = 8000 \). (B)

We could answer this question without the use of an exponential equation by noting that population grows by a factor of \( 4 = 2^2 \) in two hours, so the number of bacteria has doubled every hour. After a third hour then, the population will reach \( 2 \cdot 4000 = 8000 \).

15) For a function \( f(x) \) which is continuous on an interval \([a, b]\) , the Extreme Value Theorem guarantees that the function has a largest possible value (the absolute maximum) and a smallest possible value (the absolute minimum). These may occur at the critical points of the function where \( f'(x) = 0 \) (the local or relative maxima and minima, if any) and/or at the endpoints of the interval. We will need to examine both of these sets of possibilities.

The rational function \( f(x) = \frac{2x}{x^2 + 3} \) is always defined (since the denominator is always positive), so it is a continuous function on \textbf{any} interval. Thus, the Extreme Value Theorem certainly applies on \([-1, 3]\) . At the endpoints, the values of this function are \( f(-1) = \frac{2 \cdot (-1)}{(-1)^2 + 3} = \frac{-2}{4} = -\frac{1}{2} \) and \( f(3) = \frac{2 \cdot 3}{3^2 + 3} = \frac{6}{12} = \frac{1}{2} \).

Applying the Quotient Rule, we can locate any local extrema by calculating

\[
\begin{align*}
 f'(x) &= \frac{(x^2 + 3) \cdot (2x) - (2x) \cdot (x^2 + 3)}{(x^2 + 3)^2} \\
 &= \frac{2x^2 + 6 - 4x^2}{(x^2 + 3)^2} \\
 &= \frac{6 - 2x^2}{(x^2 + 3)^2} \\
 &= 0.
\end{align*}
\]

(continued)
Since the denominator of this ratio is never zero, we only need to consider where the numerator equals zero. We then find $6 - 2x^2 = 0 \Rightarrow 2x^2 = 6 \Rightarrow x^2 = 3 \Rightarrow x = \pm \sqrt{3}$. But $-\sqrt{3} < -1$, so there is only one local extremum in the interval, at $x = +\sqrt{3}$, for which $f(\sqrt{3}) = \frac{2\sqrt{3}}{(\sqrt{3})^2 + 3} = \frac{2\sqrt{3}}{6} = \frac{\sqrt{3}}{3} \approx 0.5774$. Because $\frac{\sqrt{3}}{3}$ is (slightly) larger than $\frac{1}{2}$, we see that $f(-1) < f(3) < f(\sqrt{3})$; hence, on the interval $[-1, 3]$, $f(-1) = -\frac{1}{2}$ is the absolute minimum and $f(\sqrt{3}) = \frac{\sqrt{3}}{3}$ is the absolute maximum (and a local maximum).

16) To answer the questions in this problem, it will be useful to find the first two derivative functions at the start:

$$f(x) = x^3 - 12x + 3 \Rightarrow f'(x) = 3x^2 - 12 \Rightarrow f''(x) = 6x.$$  

**c, a)** Because $f(x)$ is a polynomial function, it is defined and continuous “everywhere” (for all real numbers $\mathbb{R}$). The only type of critical point will then be those for which $f'(x) = 3x^2 - 12 = 0 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2$. At these points, $f''(-2) = 6(-2) < 0$ (local maximum) and $f''(2) = 6 \cdot 2 > 0$ (local minimum). We calculate that $f(-2) = (-2)^3 - 12(-2) + 3 = -8 + 24 + 3 = 19$ and $f(2) = 2^3 - 12(2) + 3 = 8 - 24 + 3 = -13$. Thus, for this function, $(-2, 19)$ is the local maximum and $(2, -13)$ is the local minimum.

We set up inequalities in order to find the intervals of increase and decrease. We find that $f'(x) = 3x^2 - 12 > 0 \Rightarrow x^2 > 4 \Rightarrow |x| > 2$ and $f'(x) < 0 \Rightarrow |x| < 2$. So $f(x)$ is increasing on the intervals $(-\infty, -2)$ and $(2, \infty)$ and decreasing over the interval $(-2, 2)$.

**b)** Inflection points in the curve represented by $f(x)$ certainly occur where $f''(x) = 0$ but $f'(x) \neq 0$ (if both derivatives are zero at a point, it may be an inflection point, but could instead be a very shallow extremum). The only place where the second derivative is zero is given by $f''(x) = 6x = 0 \Rightarrow x = 0$; plainly there, $f'(0) = -12 \neq 0$. So this function has a single inflection point at $(0, 3)$.

Concavity of the curve represented by $f(x)$ is determined by the sign of $f''(x)$. It is clear that $f''(x) = 6x$ has the same sign as the chosen value of $x$, so the curve is concave upward [$f''(x) > 0$] for $x > 0$ [the interval $(0, \infty)$] and concave downward [$f''(x) < 0$] for $x < 0$ [the interval $(-\infty, 0)$].

**d)** A graph of this function is presented in the Answer Key.

17) Once we write out this differential equation using the meaning of the exponential function with a negative exponent, it becomes clearer that this is a separable equation:

$$\frac{dy}{dx} = (x - 3)e^{-2y} \Rightarrow \frac{dy}{dx} = (x - 3) \cdot \frac{1}{e^{2y}} \Rightarrow e^{2y} \, dy = (x - 3) \, dx$$

$$\Rightarrow \int e^{2y} \, dy = \int (x - 3) \, dx \Rightarrow \frac{1}{2} e^{2y} = \frac{1}{2} x^2 - 3x + C.$$
This is usually a good point at which to insert the values of \( x \) and \( y \) from an “initial-value problem” in order to determine \( C \) -- it can be much more difficult to untangle the algebra further along. We are given that \( y(0) = \ln 2 \), so

\[
\frac{1}{2} e^{2(\ln 2)} = \frac{1}{2} \cdot 0^2 - 3 \cdot 0 + C \quad \Rightarrow \quad \frac{1}{2} e^{(\ln 2)^2} = 0 - 0 + C \quad \Rightarrow \quad \frac{1}{2} \cdot 4 = 2 = C.
\]

Hence, the specific solution function which satisfies this initial-value problem is described by \( \frac{1}{2} e^{2y} = \frac{1}{2} x^2 - 3x + 2 \). As an explicit function, we find

\[
\frac{1}{2} e^{2y} = \frac{1}{2} x^2 - 3x + 2 \quad \Rightarrow \quad e^{2y} = x^2 - 6x + 4 \quad \Rightarrow \quad \ln(e^{2y}) = \ln(x^2 - 6x + 4)
\]

\[
\Rightarrow \quad 2y = \ln(x^2 - 6x + 4) \quad \Rightarrow \quad y = \frac{1}{2} \ln(x^2 - 6x + 4) \quad \text{or} \quad \ln(\sqrt{x^2 - 6x + 4})
\]

18)  

a) Logarithmic differentiation is really just the process of taking the natural logarithm of an explicit or implicit expression before calculating the derivative. For the given explicit function, we obtain

\[
y = 2x e^{-x} (3x + 4)^5 \quad \Rightarrow \quad \ln(y) = \ln[2x e^{-x} (3x + 4)^5]
\]

\[
\Rightarrow \quad \ln y = \ln 2 + \ln x + \ln(e^{-x}) + \ln[(3x + 4)^5]
\]

\[
= \ln 2 + \ln x + (-x) + 5 \ln(3x + 4)
\]

\[
\Rightarrow \quad \frac{d}{dx}(\ln y) = \frac{d}{dx}\{\ln 2 + \ln x - x + 5 \ln(3x + 4)\}
\]

\[
\Rightarrow \quad \frac{d}{dy}(\ln y) \cdot \frac{dy}{dx} = \frac{d}{dx}(\ln 2) + \frac{d}{dx}(\ln x) - \frac{d}{dx}(x) + 5 \frac{d}{dx}[\ln(3x + 4)]
\]

\[
\Rightarrow \quad \frac{1}{y} \cdot \frac{dy}{dx} = 0 + \frac{1}{x} - 1 + 5 \frac{d}{du} (\ln u) \cdot \frac{d}{du}(3x + 4)
\]

\[
= \frac{1}{x} - 1 + 5 \cdot \left(\frac{1}{u}\right) \cdot (3) = \frac{1}{x} - 1 + \left(\frac{15}{3x + 4}\right)
\]

\[
= \frac{(3x + 4) - x (3x + 4) + 15x}{x (3x + 4)} = \frac{4 + 14x - 3x^2}{x (3x + 4)}
\]

\[
\Rightarrow \quad \frac{dy}{dx} = y \cdot \frac{4 + 14x - 3x^2}{x (3x + 4)} = \frac{2x e^{-x} (3x + 4)^5 \cdot (4 + 14x - 3x^2)}{x (3x + 4)}
\]

\[
= 2 e^{-x} (4 + 14x - 3x^2) (3x + 4)^4.
\]
b) An integrand which is a product of a polynomial times \( \ln x \) will generally require integration-by-parts:

\[
\int \frac{x \ln x}{2} \, dx = \frac{1}{2} \int x \ln x \, dx = \frac{1}{2} \left[ \left( \ln x \right) \left( \frac{1}{2} x^2 \right) - \int \frac{1}{u} \cdot \frac{1}{2} \, dx \right]
\]

set \( u = \ln x \), \( dv = x \, dx \)

\[
= \frac{1}{2} \left[ \frac{1}{2} x^2 \ln x - \frac{1}{2} \int x \, dx \right] = \frac{1}{2} \left[ \frac{1}{2} x^2 \ln x - \frac{1}{2} \left( \frac{1}{2} x^2 \right) \right] + C
\]

\[
= \frac{1}{4} x^2 \ln x - \frac{1}{8} x^2 + C.
\]

19) For this box with rectangular sides, we will call its height \( H \); it has a square base with sides of a length we will call \( L \). Its volume is then \( V = H \cdot L^2 \). Since the box has no top, it has a surface composed of the base, with area \( L^2 \), and four sides, all with area \( H \cdot L \). The total surface area of the box is thus \( L^2 + 4 \cdot H \cdot L = 192 \, \text{cm}^2 \).

If we decide to approach this as an optimization problem for a function of one variable, we must find a way to eliminate one of the variables in the volume function, which has two, \( H \) and \( L \). We can use the (constraint) equation more easily to eliminate \( H \) by re-arranging it as \( L^2 + 4HL = 192 \Rightarrow 4HL = 192 - L^2 \Rightarrow H = \frac{192 - L^2}{4L} = \frac{48}{L} - \frac{L}{4} \). The volume function becomes \( V = \left( \frac{48}{L} - \frac{L}{4} \right) L^2 = 48L - \frac{1}{4}L^3 \). We wish to find the critical point(s) of the volume function, where

\[
\frac{dV}{dL} = 0 \Rightarrow \frac{d}{dL} \left( 48L - \frac{1}{4}L^3 \right) = 48 - \frac{3}{4}L^2 = 0
\]

\[
\Rightarrow \frac{3}{4}L^2 = 48 \Rightarrow L^2 = \frac{4}{3} \cdot 48 = 64 \Rightarrow L = 8 \, \text{cm}.
\]

(We ignore the negative square-root, since \( L \) is a physical measurement.) From this, we find the height of the box to be \( H = \frac{48}{8} - \frac{8}{4} = 6 - 2 = 4 \, \text{cm} \), so its volume is \( V = 4 \cdot 8^2 = 256 \, \text{cm}^3 \).

We can confirm that this is the maximum possible volume for the box, under the constraint for surface area, by evaluating the second derivative

\[
\frac{d^2V}{dL^2} = \frac{d^2}{dL^2} \left( 48L - \frac{1}{4}L^3 \right) = \frac{d}{dL} \left( 48 - \frac{3}{4}L^2 \right) = -\frac{3}{4}L.
\]

Since \( L \) must be a positive number, the second derivative is negative, which is the indicator of a local maximum of the function \( V(L) \).
If you have covered Lagrange multipliers when you took this course, we can use them to directly optimize the volume function with its two variables, $V(H, L) = H \cdot L^2$. We write the “constraint equation” for the surface area as if it were a function being set equal to zero, that is, $L^2 + 4HL = 192 \implies S(H, L) = L^2 + 4HL - 192 = 0$. At a critical point of the function to be optimized, we can set up equations involving each first partial derivative of the function and its constraint, which use a real constant $\lambda$ (the “multiplier”):

$$
V_H = \lambda \cdot S_H \quad \rightarrow \quad \frac{\partial}{\partial H}(H L^2) = \lambda \frac{\partial}{\partial H}(L^2 + 4HL - 192) \implies L^2 = \lambda \cdot 4L \quad \text{and}
$$

$$
V_L = \lambda \cdot S_L \quad \rightarrow \quad \frac{\partial}{\partial L}(H L^2) = \lambda \frac{\partial}{\partial L}(L^2 + 4HL - 192) \implies H \cdot 2L = \lambda (2L + 4H).
$$

We now need to solve these equations simultaneously by any convenient method (there is no general technique for solving systems of such “Lagrange equations”). From the first equation, we can determine that $L^2 = \lambda \cdot 4L \implies \lambda = \frac{L}{4}$. We can then insert this result into the second equation to find

$$
H \cdot 2L = \lambda (2L + 4H) \implies 2HL = \frac{L^2}{4} (2L + 4H) = \frac{L^2}{2} + HL
$$

$$
\implies HL = \frac{L^2}{2} \implies H = \frac{L}{2}.
$$

Now that we know the relation between $H$ and $L$, we can use this in the constraint equation for the surface area of the box to calculate

$$
L^2 + 4HL = 192 \implies L^2 + 4 \cdot \left(\frac{L}{2}\right) \cdot L = 192 \implies 3L^2 = 192
$$

$$
\implies L^2 = 64 \implies L = 8 \text{ cm.} \implies H = 4 \text{ cm.},
$$

giving us the same result for the maximal box volume that we found above.

20) We are to apply the integral formula

$$
\int \frac{1}{\sqrt{a^2 + u^2}} \, du = \ln \left| u + \sqrt{a^2 + u^2} \right| + C
$$

in order to compute the definite integral

$$
\int_0^1 \frac{6}{\sqrt{3 + 9x^2}} \, dx.
$$

The symbol $a$ in the formula represents an unspecified numerical constant: for our integral, we would set its value as given by $a^2 = 3 \implies a = \sqrt{3}$ (note that although $a^2$ is used in the formula, this does not need to be a “perfect square”). Similarly, we set $u^2 = 9x^2 \implies u = 3x \implies du = 3 \, dx$. We will not really be making a “u-substitution” here, so we will not need to “transform” the limits of integration. Instead, we are able to write
\[ \int_0^1 \frac{6}{\sqrt{3 + 9x^2}} \, dx = 6 \int_0^1 \frac{1}{\sqrt{3 + 9x^2}} \left( \frac{1}{3} \, dx \right) = 2 \int_0^1 \frac{1}{\sqrt{3 + 9x^2}} \, dx \]

\[ = 2 \left( \ln \left| 3x + \sqrt{3 + 9x^2} \right| \right)
\left\|_0^1 \right. \]

\[ = 2 \left[ \left( \ln \left| 3 \cdot 1 + \sqrt{3 + 9 \cdot 1^2} \right| \right) - \left( \ln \left| 3 \cdot 0 + \sqrt{3 + 9 \cdot 0^2} \right| \right) \right] \]

\[ = 2 \left\{ \left[ \ln \left( 3 + \sqrt{12} \right) \right] - \left[ \ln \left( \sqrt{3} \right) \right] \right\} \text{ or } 2 \left\{ \left[ \ln \left( 3 + 2\sqrt{3} \right) \right] - \left[ \ln \left( \sqrt{3} \right) \right] \right\} , \]

or, using the properties of logarithms,

\[ = 2 \ln \left( \frac{3 + 2\sqrt{3}}{\sqrt{3}} \right) = \ln \left[ \left( \frac{3 + 2\sqrt{3}}{\sqrt{3}} \right)^2 \right] = \ln \left[ \frac{(3 + 2\sqrt{3})^2}{(\sqrt{3})^2} \right] \]

\[ \log_a x - \log_a y = \log_a \left( \frac{x}{y} \right) \quad p \log_a x = \log_a x^p \]

\[ = \ln \left[ \left( \frac{3^2 + 2 \cdot 3 \cdot 2\sqrt{3} + (2\sqrt{3})^2}{3} \right) \right] = \ln \left[ \frac{(9 + 12\sqrt{3} + 12)}{3} \right] = \ln \left( 7 + 4\sqrt{3} \right) . \]

G. Ruffa – 8/12