1) Since all of the choices deal with the same limit, we will attempt to evaluate \( \lim_{x \to 1} \frac{|x-1|}{x-1} \).

The presence of absolute value brackets generally indicates that we have two cases to examine: for \( x - 1 \geq 0 \) or \( x \geq 1 \), \( |x-1| = x-1 \), but for \( x - 1 < 0 \) or \( x < 1 \), \( |x-1| = -(x-1) \). This tells us then that the "right-hand" limit (or "limit from above") is

\[
\lim_{x \to 1^+} \frac{|x-1|}{x-1} = \lim_{x \to 1^+} \frac{x-1}{x-1} = 1 ,
\]

while the "left-hand" limit (or "limit from below") is

\[
\lim_{x \to 1^-} \frac{|x-1|}{x-1} = \lim_{x \to 1^-} \frac{-(x-1)}{x-1} = -1 .
\]

These "one-sided limits" do not agree, so the "two-sided limit" \( \lim_{x \to 1} \frac{|x-1|}{x-1} \) does not exist. \( \text{(D)} \)

2) The slope \( m \) of the tangent line to the curve \( y = f(x) \) at the point \( (x_0, y_0) \) is given by the value of the first derivative at \( x = x_0 \), \( m = f'(x_0) \). For the function \( f(x) = \frac{1}{x} + x \) or \( x^{-1} + x^1 \), the derivative function is \( f'(x) = (-1) x^{-2} + 1 \cdot x^0 \) or \( - \frac{1}{x^2} + 1 \). So the slope of the tangent line at \( (1, 2) \) is \( f'(1) = - \frac{1}{1^2} + 1 = 0 \).

By using the point-slope form for the equation for a line, \( (y - y_0) = m(x - x_0) \), we find that the equation of the tangent line is \( y - 2 = 0 \cdot (x - 1) \Rightarrow y = 2 \). (Since the slope is zero, the line is horizontal and thus represents a constant function.) \( \text{(A)} \)

3) The Extreme Value Theorem guarantees that a continuous function has a maximum and a minimum value somewhere on the closed interval \([a, b]\). Those may occur at local extrema (a maximum or a minimum) or at the endpoints, so we must check both possibilities.

The polynomial \( f(x) = 2x^3 - 3x^2 - 12x + 5 \) is continuous "everywhere" (over all of the real numbers), so this Theorem certainly applies on the interval \([-3, 3]\). At the endpoints, the function has the values \( f(-3) = 2(-3)^3 - 3(-3)^2 - 12(-3) + 5 = 54 - 27 + 36 + 5 = 64 \) and \( f(3) = 2(3)^3 - 3(3)^2 - 12(3) + 5 = 54 - 27 - 36 + 5 = -4 \). The local extrema are located at the values of \( x \) where \( f'(x) = 6x^2 - 6x - 12 = 0 \), \( 6(x-2)(x+1) = 0 \Rightarrow x = -1, 2 \). The values of our function at those locations are \( f(-1) = 2(-1)^3 - 3(-1)^2 - 12(-1) + 5 = -2 - 3 + 12 + 5 = 15 \) and \( f(2) = 2(2)^3 - 3(2)^2 - 12(2) + 5 = 16 - 12 - 24 + 5 = -15 \).

Among these results, \( f(-3) = 64 \) is the smallest value of \( f(x) \) in the closed interval, so it represents the absolute minimum on \([-3, 3]\); the largest value on the interval is \( f(-1) = 15 \), making it the absolute maximum. \( \text{(C)} \)
4) The complicated-looking function \( f(x) = e^x \ln x + x^{1/3} - 2 - \frac{1}{x^2 + 1} \) is the sum of four terms, so we may find its derivative function by differentiating each term in the sum in turn. The first term is a product of two functions, so we apply the product rule to find \((e^x \ln x)' = (e^x)' \cdot \ln x + e^x \cdot (\ln x)' = e^x \ln x + e^x \cdot \left(\frac{1}{x}\right)\). The next term is a power-law function, for which the derivative is \((x^{1/3})' = \frac{1}{3} x^{(1/3)-1} = \frac{1}{3} x^{-2/3}\). The third term is a constant, so its derivative is simply zero: \(2' = 0\). Finally, while the fourth term is a quotient, when the numerator is a constant, we can write the expression as \(\frac{1}{x^2 + 1} = (x^2 + 1)^{-1}\) and differentiate it as a power-law function, while also applying the Chain Rule:

\[
\frac{d}{dx} [ (x^2 + 1)^{-1} ] = \frac{d}{du} (u^{-1}) \cdot \frac{du}{dx} = (-u^{-2}) \cdot \left(\frac{2x}{x^2 + 1}\right) = -\frac{2x}{(x^2 + 1)^2} .
\]

The derivative function is thus

\[
f'(x) = e^x \ln x + \frac{e^x}{x} + \frac{1}{3} x^{-2/3} - 0 - \left(-\frac{2x}{(x^2 + 1)^2}\right) = e^x \ln x + \frac{e^x}{x} + \frac{1}{3} x^{-2/3} + \frac{2x}{(x^2 + 1)^2} .
\]

(D)

5) We are asked to find the second derivative of the function \( f(x) = \frac{x}{x + 1} \). We will first need to obtain the first derivative, which can be accomplished by the use of the Quotient Rule:

\[
f'(x) = \left(\frac{x}{x - 1}\right)' = \frac{(x-1) \cdot x' - x \cdot (x-1)'}{(x-1)^2} = \frac{x - 1 - x \cdot 1}{(x-1)^2} = \frac{-1}{(x-1)^2} .
\]

Differentiate this function in turn will give us the second derivative of \( f(x) \):

\[
f''(x) = [f(x)]'' = \left[\frac{-1}{(x-1)^2}\right]' = \left[-\frac{1}{x-1}\right]' = -\frac{d}{du} (u^{-2}) \cdot \frac{du}{dx} = -\left(-2u^{-3}\right) \cdot \frac{du}{dx} \cdot (x-1) = \left(2u^{-3}\right) \cdot 1 = 2(x-1)^{-3} \quad \text{or} \quad \frac{2}{(x-1)^3} .
\]

(C)

6) Since the derivative of \( \ln x \) is \( \frac{1}{x} \), we can carry out this integration by “u-substitution”, for which we have \( u = \ln x \) and \( du = \frac{1}{x} \, dx \). The limits for the definition integral becomes \( x = 1 \rightarrow u = \ln 1 = 0 \) and \( x = 3 \rightarrow u = \ln 3 \), and the integral itself is

\[
\int_{\ln 1}^{\ln 3} \frac{\ln x}{x} \, dx = \int_{1}^{3} \frac{\ln x}{u} \cdot \frac{1}{x} \, du \rightarrow \int_{0}^{\ln 3} u \, du = \frac{1}{2} u^2 \bigg|_{0}^{\ln 3} = \frac{1}{2} (\ln 3)^2 - \frac{1}{2} (0)^2 = \frac{1}{2} (\ln 3)^2 .
\]

(B)
7) The average value of a function \( f(x) \) on the interval \([a, b]\) is given by 
\[
\langle f(x) \rangle_{[a,b]} = \frac{\int_a^b f(x) \, dx}{b - a}.
\]
So the average value of \((x - 1)\) over the interval \([0, 2]\) is calculated (using a \(u\)-substitution) from
\[
\int_0^2 (x - 1)^3 \, dx \quad \text{to} \quad \int_{-1}^1 u^3 \, du
\]
\[
\frac{2}{2 - 0} \rightarrow \frac{1}{2} = \frac{1}{2} \left( \frac{1}{4} u^4 \right) \bigg|_{-1}^{1} = \frac{1}{8} \left( (1)^4 - (-1)^4 \right) = \frac{1}{8} \left( 1 - 1 \right) = 0 . \quad \text{(B)}
\]

8) We evaluate this (Type I) improper integral by choosing a variable for the upper limit and then allowing the limit to “run to infinity”. Because the integrand is an exponential function \(e^{kx}\), we need to make a substitution, since we really only know the antiderivative for \(e^{x}\):
\[
\int_0^\infty 5e^{-5x} \, dx = \lim_{t \to \infty} 5 \int_0^t e^{-5x} \, dx \quad \rightarrow \quad \lim_{t \to \infty} 5 \int_0^t e^{u} \left(-\frac{1}{5} \, du\right) = \lim_{t \to \infty} \int_0^t e^{u} \, du
\]
\[
= \lim_{t \to \infty} (-e^u) \bigg|_0^t = \lim_{t \to \infty} (-e^t) - (-e^0)
\]
\[
= 0 - (-1) = 1 . \quad \text{(B)}
\]

9) In partial differentiation, the rules for derivatives are the same as in ordinary differentiation, but we treat as constants \textit{all variables other than the one} that is the basis for the differentiation. We are asked to find a second partial derivative of a function, so we must apply this process twice:

\[
f(x, y) = (x^2 + y^2) e^{xy}:
\]
\[
\frac{\partial f}{\partial y} = \frac{\partial}{\partial y} \left( (x^2 + y^2) e^{xy} \right) = \frac{\partial}{\partial y} (x^2 + y^2) \cdot e^{xy} + (x^2 + y^2) \cdot \frac{\partial}{\partial y} e^{xy}
\]
\[
= 2y \cdot e^{xy} + (x^2 + y^2) \cdot e^{xy} = e^{xy} \left[ 2y + (x^2 + y^2) \cdot x \right]
\]
\[
= e^{xy} (2y + x^3 + xy^2) ;
\]
\[
\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial y} \right) = \frac{\partial}{\partial x} \left[ e^{xy} (2y + x^3 + xy^2) \right]
\]
\[
= \frac{\partial}{\partial x} (e^{xy}) \cdot (2y + x^3 + xy^2) + e^{xy} \cdot \frac{\partial}{\partial x} (2y + x^3 + xy^2)
\]
\[
= (ye^{xy}) \cdot (2y + x^3 + xy^2) + e^{xy} \cdot (0 + 3x^2 + y^2)
\]
\[
= e^{xy} \cdot (2y^2 + x^3y + x^2y + 3x^2 + y^2)
\]
\[
= e^{xy} \cdot (3x^2 + 3y^2 + x^3y + xy^3) . \quad \text{(A)}
\]
10)  

a) The critical points of a function $f(x)$ occur at the values of $x$ for which $f'(x)$ is either zero or undefined. Since $f(x) = x^4 - 4x^3 + 12$ is a polynomial, its derivatives are always defined, so we wish to find where

$$f'(x) = 4x^3 - 12x^2 = 0 \implies 4x^2(x - 3) = 0 \implies x = 0, 3.$$  

So the critical points are $x = 0$ and $x = 3$.

b) The second derivative $f''(x)$ provides us with information about concavity of the curve for the function, which in turn will help us to identify the types of critical points; points where $f''(x) = 0$ and $f'(x) \neq 0$ are inflection points of the curve. (A point at which both $f''(x)$ and $f'(x)$ are zero may be an inflection point, but could instead be a local extremum.)

We find that

$$f''(x) = 12x^2 - 24x = 0 \implies 12x(x - 2) = 0 \implies x = 0, 2.$$  

The special points for this function can now be characterized as follows:

- $x = 3$ : $f'(3) = 0$, $f''(3) = 12 \cdot 3^2 - 24 \cdot 3 = 108 - 72 > 0$, so $x = 3$ is a **local minimum**.
- $x = 2$ : $f'(2) \neq 0$, $f''(2) = 0$, so $x = 2$ is an **inflection point**.
- $x = 0$ : $f'(0) = 0$ and $f''(0) = 0$, which requires us to examine how the concavity behaves in the neighborhood of this point; for values of $x$ slightly smaller than zero, both $x$ and $x - 2$ are negative, so $12x(x - 2)$ is a positive product there, but for values of $x$ slightly larger than zero, $x$ is positive but $x - 2$ is still negative, so $12x(x - 2)$ becomes negative; thus, the concavity of the curve does change at $x = 0$, making it an **inflection point**.

c, d) With the special points of the function located, we can investigate the behavior of the first and second derivatives in the intervals surrounding this points:

<table>
<thead>
<tr>
<th></th>
<th>$x &lt; 0$</th>
<th>$0 &lt; x &lt; 2$</th>
<th>$2 &lt; x &lt; 3$</th>
<th>$x &gt; 3$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>sign of</strong> $f'(x)$</td>
<td>$\cdot + \cdot - = \cdot - \cdot - = \cdot + \cdot - = \cdot -$</td>
<td>$\cdot - \cdot - = \cdot - \cdot - = \cdot + \cdot - = \cdot +$</td>
<td>$\cdot - \cdot - = \cdot - \cdot - = \cdot + \cdot - = \cdot +$</td>
<td>$\cdot - \cdot - = \cdot - \cdot - = \cdot + \cdot - = \cdot +$</td>
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</tr>
</tbody>
</table>

This shows us that $f(x)$ decreases for $x < 0$ and $0 < x < 3$ and increases for $x > 3$; the curve is concave upward for $x < 0$ and $2 < x < 3$ and concave downward for $0 < x < 2$.

e) A graph of $f(x)$ is presented in the Answer Key.
11)

a) While the equation \( x^2 - 3xy + y^3 = -1 \) could in principle be solved explicitly for a function \( y = f(x) \), the expression would be a bit complicated and not especially simple to differentiate. With the understanding that such a function exists, we can calculate its derivative implicitly to arrive at a description of the derivative’s behavior; we achieve this through the use of the Chain Rule:

\[
\frac{d}{dx}(x^2 - 3xy + y^3) = \frac{d}{dx}(-1) \Rightarrow \frac{d}{dx}(x^2) - 3\frac{d}{dx}(xy) + \frac{d}{dx}(y^3) = 0
\]

\[
\Rightarrow 2x - 3\left[ x \cdot \frac{dy}{dx} + y \cdot \frac{dx}{dx} \right] + 3y^2 \cdot \frac{dy}{dx} = 0 \Rightarrow 2x - 3y - 3x\frac{dy}{dx} + 3y^2 \frac{dy}{dx} = 0
\]

\[
\Rightarrow (3y^2 - 3x) \frac{dy}{dx} = 3y - 2x \Rightarrow \frac{dy}{dx} = \frac{3y - 2x}{3y^2 - 3x}
\]

b) The equation of the tangent line at a point on the curve described by this equation is found in the same way as we did in Problem 2 above; however, for an implicit function, it is often necessary to specify both coordinates of the point at which we wish to find the tangent line. The slope \( m \) of the tangent line to this curve at \((2, 1)\) is found from

\[
m = \left. \frac{dy}{dx} \right|_{(2,1)} = \frac{3 \cdot 1 - 2 \cdot 2}{3 \cdot 1^2 - 3 \cdot 2} = \frac{3 - 4}{3 - 6} = \frac{-1}{-3} = \frac{1}{3}
\]

The point-slope form for the equation of the tangent line is then \( y - 1 = \frac{1}{3}(x - 2) \); in slope-intercept form, this becomes \( y - 1 = \frac{1}{3}x - \frac{2}{3} \Rightarrow y = \frac{1}{3}x + \frac{1}{3} \).

12) We are asked to find the area under a curve (and above the \( x \)-axis) which extends infinitely to the right, so we must carry out an improper integration; as we did in Problem 8, we compute this result by taking a “limit at infinity”:

\[
\int_0^\infty xe^{-x^2} \, dx = \lim_{t \to \infty} \int_0^t xe^{-x^2} \, dx \quad \Rightarrow \quad \lim_{t \to \infty} \int_0^t e^{-u} \left( \frac{1}{2} \, du \right)
\]

\[
= \lim_{t \to \infty} \frac{1}{2} \int_0^t e^{-u} \, du = \lim_{t \to \infty} \frac{1}{2} \left[ -e^{-u} \right]_0^t = -\frac{1}{2} \left[ \lim_{t \to \infty} (e^{-t}) - (e^0) \right]
\]

\[
= -\frac{1}{2} \left[ 0 - 1 \right] = \frac{1}{2}
\]
13) The velocity function \( v(t) \) of an object is found as the derivative with respect to time \( t \) of the object’s position function \( x(t) \). For the position function given, we find

\[
v(t) = \frac{d}{dt} x(t) = \frac{d}{dt} \left[ 360,000 - e^{t/10} \cdot (360,000 - 36,000t + 1800t^2 - 60t^3 + t^4) \right]
\]

\[
= \frac{d}{dt} 360,000 - \left[ \frac{d}{dt} e^{t/10} \cdot (360,000 - 36,000t + 1800t^2 - 60t^3 + t^4) \right] \\
+ e^{t/10} \cdot \frac{d}{dt} (360,000 - 36,000t + 1800t^2 - 60t^3 + t^4)
\]

\[
= 0 - \left[ \frac{d}{du} e^u \cdot \frac{du}{dt} \right] \cdot (360,000 - 36,000t + 1800t^2 - 60t^3 + t^4) \\
+ e^{t/10} \cdot (0 - 36,000 + 3600t - 180t^2 + 4t^3)
\]

\[
= -\left[ e^u \cdot \frac{d}{dt} \left[ \frac{t}{10} \right] \right] \cdot (360,000 - 36,000t + 1800t^2 - 60t^3 + t^4) \\
+ e^{t/10} \cdot (-36,000 + 3600t - 180t^2 + 4t^3)
\]

\[
= -e^{t/10} \cdot \left[ (360,000 - 36,000t + 1800t^2 - 60t^3 + \frac{1}{10}t^4) + (-36,000 + 3600t - 180t^2 + 4t^3) \right]
\]

\[
= -e^{t/10} \cdot (-2t^3 + \frac{1}{10}t^4) = t^3 e^{t/10} (2 - \frac{1}{10}t) .
\]

At \( t = 0 \), the velocity of this object is \( v(0) = 0^3 \cdot e^{0/10} (2 - \frac{1}{10} \cdot 0) = 0 \cdot e^0 \cdot 2 = 0 \).

For times after \( t > 0 \), the factors \( t^3 \) and \( e^{t/10} \) are positive (in fact, \( e^{t/10} \) is always positive); as for the factor \( (2 - \frac{1}{10}t) \), \( 2 - \frac{1}{10}t > 0 \) \( \Rightarrow \) \( 2 > \frac{1}{10}t \) \( \Rightarrow \) \( 20 > t \).

So all three factors appearing in the velocity function are positive for \( 0 < t < 20 \), and thus \( v(t) \) is certainly positive (object moves to the right) for \( 0 < t \leq 10 \).

14) The topic to which this Problem is related is no longer included in this course, but it will be worthwhile to present the calculations, since they are a good exercise in definite integration.

a) The mean of the probability density function \( f(x) = \frac{x^4}{625} \) on the interval \([0 , 5]\) is found by a computation similar to the one we used in Problem 7, except that we wish to find the “weighted average” for \( x \) by integrating \( x \cdot f(x) \) over the interval:

\[
\mu = \{ x \}_{0.5} = \int_0^5 x \cdot \frac{x^4}{625} \, dx = \frac{1}{625} \int_0^5 x^5 \, dx = \frac{1}{625} \left( \frac{1}{6} x^6 \right) \bigg|_0^5
\]
\[ = \frac{1}{6 \cdot 5^4} \left[ (5^6) - (0^6) \right] = \frac{5^6}{6 \cdot 5^4} = \frac{25}{6}. \]

**b)** To find the standard deviation of the probability density function, we will need first to compute its variance \( V \), which is related to the weighted average of \( x \) on the interval, through the equation

\[
V = \left\langle x^2 \right\rangle_{[0,5]} - \mu^2 = \left[ \int_0^5 x^2 \cdot \frac{x^4}{625} \, dx \right] - \left( \frac{25}{6} \right)^2 = \left[ \frac{1}{625} \int_0^5 x^6 \, dx \right] - \left( \frac{25}{6} \right)^2
\]

\[
= \left[ \frac{1}{625} \left( \frac{1}{7} x^7 \right) \right]_0^5 - \left( \frac{5^4}{6^2} \right)^2 = \left\{ \frac{1}{7 \cdot 5^4} \left[ (5^7) - (0^7) \right] \right\} - \left( \frac{5^4}{6^2} \right)
\]

\[
= \left( \frac{5^7}{7 \cdot 5^4} \right) - \left( \frac{5^4}{6^2} \right) = \left( \frac{5^3}{7} \right) - \left( \frac{5^4}{6^2} \right) = \frac{5^3 \cdot 6^2 - 5^4 \cdot 7}{7 \cdot 6^2} = \frac{5^3 \cdot (36 - 5 \cdot 7)}{7 \cdot 6^2}
\]

\[
= \frac{5^3 \cdot (36 - 35)}{7 \cdot 6^2} = \frac{5^3 \cdot 1}{7 \cdot 6^2} = \frac{5 \cdot 5^2}{7 \cdot 6^2} \text{ or } \frac{125}{252}.
\]

The standard deviation \( \sigma \) for this function on \([0,5]\) is the square-root of the variance on the interval, or

\[
\sigma = \sqrt{\frac{5 \cdot 5^2}{7 \cdot 6^2}} = \sqrt{\frac{5}{7}} \cdot \frac{5}{6} \text{ or } \frac{5\sqrt{35}}{42} \approx 0.7043.
\]

15) [This problem involves the standard normal distribution for probability and statistics, which is no longer covered in this course.]