1. Since the angle $\alpha$ is in the second quadrant, we know that $\sin \alpha > 0$. We are given that $\cos \alpha = -\frac{5}{\sqrt{14}}$, so we know the ratio of the horizontal leg to the hypotenuse for the right triangle shown. If we call the vertical leg $y$, which must be positive, we find

$$(-5)^2 + y^2 = 5^2 \Rightarrow y^2 = 25 - 56 \Rightarrow y = \sqrt{56} = \sqrt{4 \cdot 14} = 2\sqrt{14}.$$ 

We now have everything we need to find all of the other trigonometric ratios for $\alpha$:

$$\sin \alpha = \frac{\text{vertical leg}}{\text{hypotenuse}} = \frac{2\sqrt{14}}{5}; \quad \tan \alpha = \frac{\text{vertical leg}}{\text{horizontal leg}} = \frac{\sin \alpha}{\cos \alpha} = \frac{2\sqrt{14}}{-5}; \quad \cot \alpha = \frac{1}{\tan \alpha} = \frac{-5}{2\sqrt{14}}; \quad \sec \alpha = \frac{1}{\cos \alpha} = -\frac{5}{\sqrt{14}}; \quad \csc \alpha = \frac{1}{\sin \alpha} = \frac{9}{2\sqrt{14}}.$$ 

2. a) The composite function $\sin(\sin^{-1}(-\frac{1}{2}))$ evaluates the sine of some angle $\alpha = \sin^{-1}(-\frac{1}{2})$. This tells us that $\alpha$ is below the x-axis ($\pi < \alpha < 2\pi$) and that $\sin \alpha = -\frac{1}{2}$; it is not necessary to know the value of $\alpha$. The composite is $\sin(\sin^{-1}(-\frac{1}{2})) = \sin \alpha = -\frac{1}{2}$, which is also what we expect from applying a function to its inverse function.

b) This composite function shows the importance of giving attention to the ranges of functions. The inverse cosine function takes a value $x$ in the domain $-1 \leq x \leq 1$ and gives the angle $y$, having that value of $x$ for its cosine, in the range $0 \leq y \leq \pi$. The value $x$ here is $x = \cos \frac{3\pi}{4} = -\frac{\sqrt{2}}{2}$, but the appropriate angle in the range of the inverse cosine function is $y = \cos^{-1}(-\frac{\sqrt{2}}{2}) = \frac{3\pi}{4}$. So the result for our composite function is $\cos^{-1}(\cos(\frac{3\pi}{4})) = \frac{3\pi}{4}$.

c) The composite function $\tan(\cos^{-1}(\frac{3}{5}))$ evaluates the tangent of some angle $\beta = \cos^{-1}(\frac{3}{5})$. It is not necessary to find the value of $\beta$; we just need to know the relationships among the sides of a right triangle including the angle $\beta$. Since $\cos \beta = \frac{3}{5}$, we can set up a right triangle for which we can find the length $y$ opposite to angle $\beta$:

$$8^2 = 5^2 + y^2 \Rightarrow y^2 = 64 - 25 = 39 \Rightarrow y = \sqrt{39}.$$ 

The tangent of angle $\beta$ is thus $\tan \beta = \frac{\sqrt{39}}{5}$, so $\tan(\cos^{-1}(\frac{3}{5})) = \frac{\sqrt{39}}{5}$. Since the arccosine function has the range $0 \leq \beta \leq \pi$, $\beta$ must be in the first quadrant, so this is the only solution.
The basic function we will work with is 
\[ y = \sin x. \]

Amplitude: 1  
Period: \( 2\pi \)  
Phase shift: 0

The transformation to 
\[ y = \sin \left( \frac{\pi}{3}(x+2) \right) \]

shifts the graph horizontally to the left by 2 units.

Amplitude: 1  
Period: \( 2\pi \)  
Phase shift: \(-2\) radians

The transformation to 
\[ y = 4 \sin \left( \frac{\pi}{3}x + \frac{2\pi}{3} \right) \]

compresses the graph horizontally by a factor of \( \frac{\pi}{3} \) about the new symmetry axis at \( x = -2 \).

Amplitude: 1  
Period: \( \frac{2\pi}{(\frac{\pi}{3})} = 6 \)  
Phase shift: \(-2\)

The transformation to 
\[ y = 4 \sin \left( \frac{\pi}{3}x + \frac{2\pi}{3} \right) \]

stretches the graph vertically by a factor of 4.

Amplitude: 4  
Period: 6  
Phase shift: \(-2\)

The transformation to 
\[ y = -4 \sin \left( \frac{\pi}{3}x + \frac{2\pi}{3} \right) \]

reverses the direction of the \( y \)-coordinate, producing an inversion of the graph.

Amplitude: 4  
Period: 6  
Phase shift: \(-2\)
a) There are two ways we can evaluate this. One is to recognize that \( \frac{11\pi}{12} \) can be written as the sum of two angles for which we know trigonometric ratios: \( \frac{11\pi}{12} = \frac{3\pi}{4} + \frac{\pi}{3} \). We can then use the identity for the sum of two angles:

\[
\cos \left( \frac{11\pi}{12} \right) = \cos \left( \frac{\pi}{4} + \frac{2\pi}{3} \right) = \cos \frac{\pi}{4} \cos \frac{2\pi}{3} - \sin \frac{\pi}{4} \sin \frac{2\pi}{3}
\]

\[
= \left( \frac{\sqrt{2}}{2} \right) \left( \frac{1}{2} \right) - \left( \frac{\sqrt{2}}{2} \right) \left( \frac{\sqrt{3}}{2} \right)
\]

\[
= -\frac{\sqrt{2}}{4} - \frac{\sqrt{6}}{4} = -\frac{\sqrt{2}}{4} \cdot (1 + \sqrt{3})
\]

Alternative, we could use the half-angle formula for cosine, since \( \frac{11\pi}{12} = \frac{1}{2} \left( \frac{11\pi}{6} \right) \) and \( \frac{11\pi}{6} \) is an angle we know the trigonometric ratios for. So,

\[
\cos \left( \frac{\pi}{12} \right) = \cos \left( \frac{11\pi}{24} \right) = -\sqrt{\frac{1 + \cos (\frac{11\pi}{6})}{2}} = -\sqrt{\frac{1 + \frac{\sqrt{3}}{2}}{2}} = -\sqrt{\frac{2 + \sqrt{3}}{4}} = -\frac{\sqrt{2 + \sqrt{3}}}{2};
\]

the negative square root is used because \( \frac{11\pi}{12} \) is in the second quadrant, where \( \cos \theta < 0 \).

This two results actually describe the same number. We can write the first answer as

\[
-\frac{\sqrt{2}}{4} \cdot (1 + \sqrt{3}) = -\frac{\sqrt{2} \cdot \sqrt{(1 + \sqrt{3})^2}}{4} = -\frac{\sqrt{2} \cdot \sqrt{1 + 2\sqrt{3} + 3}}{4} = -\frac{\sqrt{2} \cdot \sqrt{4 + 2\sqrt{3}}}{4} = -\frac{\sqrt{2 + \sqrt{3}}}{2},
\]

\[
= -\frac{\sqrt{4(2 + \sqrt{3})}}{4} = -\frac{2\sqrt{2 + \sqrt{3}}}{2} = -\frac{\sqrt{2 + \sqrt{3}}}{2},
\]
giving our second result.

b) The important point to recognize in this problem is that the expression

\[\sin 82^\circ \cos 37^\circ - \cos 82^\circ \sin 37^\circ\]

resembles the expression for the sine of the difference of two angles, \( \sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta \). So our expression is equivalent to \( \sin (82^\circ - 37^\circ) = \sin 45^\circ = \frac{\sqrt{2}}{2} \).

\[\]

(5)

\[\]

a) \[\frac{1 + \cos \theta + \sin \theta}{1 + \cos \theta - \sin \theta} = \frac{1 + \cos \theta + \sin \theta}{1 + \cos \theta - \sin \theta} = \frac{1 - \cos \theta + \sin \theta}{1 - \cos \theta - \sin \theta} = \frac{1 + 2 \sin \theta - \cos^2 \theta + \sin^2 \theta}{1 + 2 \sin \theta \cos \theta - \cos^2 \theta - \sin^2 \theta} = \frac{1 + 2 \sin \theta - (1 - \sin^2 \theta) + \sin^2 \theta}{1 + 2 \sin \theta \cos \theta - (\cos^2 \theta + \sin^2 \theta)}
\]

\[= \frac{2 \sin \theta + 2 \sin^2 \theta}{2 \sin \theta \cos \theta} = \frac{1 + \sin \theta}{\cos \theta} = \sec \theta + \tan \theta\]
b) \[
\frac{\cos (a + b)}{\cos (a - b)} = \frac{\cos a \cos b - \sin a \sin b}{\cos a \cos b + \sin a \sin b} = \frac{1}{\frac{\cos a \cos b}{\cos a \cos b}} = \frac{1}{\frac{1}{\cos a \cos b}} = \frac{\cos a \cos b}{\cos a \cos b} = 1
\]
\[
1 - \tan a \tan b = \tan a \tan b
\]

\[
\sin (4\theta) = \sin (2\cdot 2\theta) = 2 \sin (2\theta) \cos (2\theta) = 2 \sin (2\theta) \cdot (\cos^2 \theta - \sin^2 \theta)
\]
\[
= 2 \sin (2\theta) \cdot [(1 - \sin^2 \theta) - \sin^2 \theta]
\]
\[
= 2 \sin (2\theta) \cdot (1 - 2 \sin^2 \theta)
\]

a) To solve this equation, we need to ask what angles have a cosine of \(\frac{1}{2}\). (Note that, in solving trigonometric equations, we are not restricted to the domains of the inverse trigonometric functions.) So we have

\[
\cos \left(\frac{\theta - \pi}{4}\right) = \frac{1}{2} \Rightarrow \frac{\theta - \pi}{4} = \frac{\pi}{3} \Rightarrow \frac{\theta}{3} = \frac{\pi}{3} + \frac{\pi}{4} = \frac{7\pi}{12}
\]

or \(\frac{\theta - \pi}{4} = \frac{5\pi}{3} \Rightarrow \frac{\theta}{3} = \frac{5\pi}{3} + \frac{\pi}{4} = \frac{23\pi}{12}
\]

The second of these results is outside the principal circle \([0, 2\pi]\), so we will need to subtract a multiple of \(2\pi\) in order to find the angle it represents within that circle:

\[
\theta = \frac{23\pi}{12} - 4\pi = \frac{29\pi - 48\pi}{12} = \frac{7\pi}{12}
\]

This is identical to our first result, so this equation only has the one solution, \(\theta = \frac{7\pi}{12}\), within the principal circle.

b) Trigonometric equations of the form \(A \sin \theta + B \cos \theta = C\) can be solved by using a method based on the angle-addition identity \(\sin (a + b) = \cos a \sin b + \sin a \cos b\).

If we put the right-hand side of this equation into correspondence with our equation, we have

\[
\frac{\cos a \sin b + \sin a \cos b}{\sqrt{A^2 + B^2}} = \frac{\sin (a + b)}{\sqrt{A^2 + B^2}} = \frac{1}{\cos \theta} = \tan a \tan b
\]

To complete the analogy, we use the Pythagorean identity, \(\sin^2 \theta + \cos^2 \theta = 1\). We divide our equation through by \(\sqrt{A^2 + B^2}\) and solve for \(a + b\) to get \(a + b = \sin (\theta + \phi)\).

So we have \(\cos a = \frac{\sqrt{2}}{2}\) and \(\sin a = \frac{\sqrt{2}}{2}\), which tells us that \(a = \frac{\pi}{4}\).

We now need to solve \(\sin (b + \frac{\pi}{2}) = \frac{1}{2}\), which can be done similarly to the way we solved the equation in part (a) above:

\[
\sin (\theta + \frac{\pi}{2}) = \frac{1}{2} \Rightarrow \theta + \frac{\pi}{2} = \frac{\pi}{6} \Rightarrow \theta = 0
\]

or \(\theta + \frac{\pi}{2} = \frac{5\pi}{6} \Rightarrow \theta = \frac{2\pi}{3} = \frac{4\pi}{6}\)

The next possibility would be \(\theta + \frac{\pi}{2} = \frac{13\pi}{6} \Rightarrow \theta = 2\pi\), which is outside the principal circle (and is redundant, since \(2\pi\) is the same direction as \(0\)). So the two solutions to our equation within the principal circle are \(\theta = 0\) and \(\theta = \frac{4\pi}{6} = \frac{2\pi}{3}\).
c) In solving a trigonometric equation, it is important to have the same argument in all of the terms. Since $4\theta$ is twice $2\theta$, this suggests that we might start by applying a double-angle identity to the $\cos 4\theta$ term. If we use
\[
\cos 2x = \cos^2 x - \sin^2 x \quad \text{with} \quad x = 2\theta,
\]
we can write
\[
\cos 2\theta + \cos 4\theta = \cos 2\theta + (\cos^2 2\theta - \sin^2 2\theta) = 0.
\]
This alone isn’t enough, as it will still be difficult to deal with the equation while it has both $\cos^2 \theta$ and $\sin^2 \theta$ terms. If we now apply the Pythagorean identity, giving $\sin^2 2\theta = 1 - \cos^2 2\theta$, our equation becomes
\[
\cos 2\theta + \cos^2 2\theta - (1 - \cos^2 2\theta) = 0 \Rightarrow 2 \cos^2 2\theta + \cos 2\theta - 1 = 0.
\]
This is now a quadratic equation to be solved – this is easier to see if we make the substitution $t = \cos 2\theta$:
\[
2t^2 + t - 1 = 0 \quad \Rightarrow \quad (2t-1)(t+1) = 0 \quad \Rightarrow \quad t = \frac{1}{2} \quad \text{or} \quad t = -1.
\]
When we “back-substitute” $\cos 2\theta$ for $t$, we obtain two trigonometric equations that we can solve more easily:
\[
t = \frac{1}{2} \quad \Rightarrow \quad \cos 2\theta = \frac{1}{2} \quad \Rightarrow \quad 2\theta = \frac{\pi}{3} \quad \Rightarrow \quad \theta = \frac{\pi}{6}
\]
or $2\theta = \frac{5\pi}{3} \Rightarrow \theta = \frac{5\pi}{6}$
or $t = -1 \quad \Rightarrow \quad \cos 2\theta = -1 \quad \Rightarrow \quad 2\theta = \pi \Rightarrow \theta = \frac{\pi}{2}$.

Bear in mind that with trigonometric equations involving multiples of $\theta$ ($n\theta$ with $n>1$), it may be possible to find more solutions by going beyond the principal circle:
\[
\cos 2\theta = \frac{1}{2} \quad \Rightarrow \quad 2\theta = \frac{\pi}{3} + 2\pi = \frac{7\pi}{3} \quad \Rightarrow \quad \theta = \frac{7\pi}{6}
\]
or $2\theta = \frac{5\pi}{3} + 2\pi = \frac{11\pi}{3} \Rightarrow \theta = \frac{11\pi}{6}$.
\[
\cos 2\theta = -1 \quad \Rightarrow \quad 2\theta = \pi + 2\pi = 3\pi \Rightarrow \theta = \frac{3\pi}{2}.
\]

The complete set of solutions for our equation is then $\theta = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{3\pi}{2}$, and $\frac{11\pi}{6}$.
To find the cosines of the angles of a triangle
for which all three sides are known, we can pretty much
start anywhere. We will use the Law of Cosines to find
the cosine of each angle in turn:

$$6^2 = 10^2 + 13^2 - 2 \cdot 10 \cdot 13 \cdot \cos \alpha$$

$$\Rightarrow \cos \alpha = \frac{10^2 + 13^2 - 6^2}{2 \cdot 10 \cdot 13} = \frac{100 + 169 - 36}{260} = \frac{233}{260}$$

$$13^2 = 10^2 + 6^2 - 2 \cdot 10 \cdot 6 \cdot \cos \beta$$

$$\Rightarrow \cos \beta = \frac{10^2 + 6^2 - 13^2}{2 \cdot 10 \cdot 6} = \frac{100 + 36 - 169}{120} = \frac{-33}{120} = -\frac{11}{40}$$

$$10^2 = 6^2 + 13^2 - 2 \cdot 6 \cdot 13 \cdot \cos \gamma$$

$$\Rightarrow \cos \gamma = \frac{6^2 + 13^2 - 10^2}{2 \cdot 6 \cdot 13} = \frac{36 + 169 - 100}{156} = \frac{105}{156} = \frac{35}{52}$$

The area of this triangle can be found by applying Heron's formula,

$$A = \sqrt{s(s-a)(s-b)(s-c)}$$

where $$s = \frac{a+b+c}{2}$$, a, b, and c being
the lengths of the triangle's sides. Thus, for our triangle,

$$s = \frac{13 + 10 + 6}{2} = \frac{29}{2} \Rightarrow A = \sqrt{\left(\frac{29}{2}\right)\left(\frac{29}{2}-13\right)\left(\frac{29}{2}-10\right)\left(\frac{29}{2}-6\right)}$$

$$= \sqrt{\frac{29}{2} \cdot \frac{3}{2} \cdot \frac{9}{2} \cdot \frac{17}{2}} = \sqrt{\frac{13311}{16}} = \frac{\sqrt{13311}}{4} \approx 28.84$$

8 The highway diversion may be divided up
into three sections. The red segment is the
hypotenuse of a right triangle, a second angle
of which is the 30° by which the road changes
direction (the supplement of 150°). Since the
distance opposite the 30° angle is 2 miles,
the hypotenuse is given by

$$\sin 30° = \frac{2}{x} \Rightarrow \frac{1}{2} \Rightarrow x = 4 \text{ miles}$$

the distance along the original direction of the road is given by

$$\cos 30° = \frac{a}{2} \Rightarrow \frac{\sqrt{3}}{2} \Rightarrow a = 2\sqrt{3} \text{ miles.}$$

By a similar argument, the blue segment is also the hypotenuse of a
right triangle containing a 45° angle (the supplement of 135°), the length of which is given by

$$\sin 45° = \frac{2}{y} \Rightarrow \frac{\sqrt{2}}{2} \Rightarrow y = 2\sqrt{2} \text{ miles.}$$

The distance along the original direction of the road
given by

$$\cos 45° = \frac{b}{2\sqrt{2}} \Rightarrow \frac{\sqrt{2}}{2} \Rightarrow b = 2 \text{ miles.}$$

This leaves the green segment of the diversion, which runs parallel to the original direction
of the road. Its length is then given by

$$8 - a - b = 8 - 2\sqrt{3} - 2 = 6 - 2\sqrt{3} \text{ miles.}$$

The total length of pavement needed for the diversion is then

$$x + (6 - 2\sqrt{3}) + y$$

$$= 4 + 6 - 2\sqrt{3} + 2\sqrt{2} = 10 - 2(\sqrt{3} - \sqrt{2}) \approx 9.38 \text{ miles.}$$
We wish to find the set of complex zeroes for the polynomial \( f(x) = 2x^4 + x^3 - 35x^2 - 113x + 65 \).

It will be of value to apply Descartes' Rule of Signs first:

\[
f(x) = 2x^4 + x^3 - 35x^2 - 113x + 65,
\]

\[
\uparrow \quad \text{sign change} \quad \uparrow \quad \text{sign change}
\]

\[
f(-x) = 2(-x)^4 + (-x)^3 - 35(-x)^2 - 113(-x) + 65
\]

\[
= 2x^4 - x^3 - 35x^2 + 113x + 65
\]

\[
\uparrow \quad \text{sign change} \quad \uparrow \quad \text{sign change}
\]

The two sign changes among the coefficients of \( f(x) \) tell us that \( f(x) \) has an even number of positive real zeroes, either zero or two. Likewise, the two sign changes among the coefficients of \( f(-x) \) mean that \( f(x) \) also has either zero or two negative real zeroes.

Since this is a fourth-degree polynomial, \( f(x) \) has at most four complex zeroes; so it could turn out that all of them are real and none are complex.

We can next turn to the Rational Zeros Theorem for candidates. Possible rational zeroes are constructed as \( \pm \frac{\text{divisor of constant term}}{\text{divisor of leading coefficient}} \). For our polynomial,

the candidates are:

\[
\pm \frac{1}{1} = \pm 1, \quad \pm \frac{1}{2} = \pm \frac{1}{2}, \quad \pm \frac{5}{1} = \pm 5, \quad \pm \frac{5}{2} = \pm \frac{5}{2}, \quad \pm \frac{13}{1} = \pm 13, \quad \pm \frac{13}{2} = \pm \frac{13}{2}, \quad \pm \frac{65}{1} = \pm 65, \quad \pm \frac{65}{2}.
\]

As we work our way along this list, we find that

\[
f(1) = 2 + 1 - 35 - 113 + 65 \neq 0,
\]

\[
f(-1) = 2 - 1 - 35 + 113 + 65 \neq 0,
\]

\[
f\left(\frac{1}{2}\right) = 2 \left(\frac{1}{2}\right)^4 + \left(\frac{1}{2}\right)^3 - 35 \left(\frac{1}{2}\right)^2 - 113 \left(\frac{1}{2}\right) + 65 = \frac{2}{16} + \frac{1}{8} - \frac{35}{4} - \frac{113}{2} + 65 = 0
\]

So \( x = \frac{1}{2} \) is a positive real zero of \( f(x) \); since there is one, there must be a second one.

We first divide \( f(x) \) by the factor \((x - \frac{1}{2})\):

\[
\begin{array}{c|ccccc}
 & 2x^3 + 2x^2 - 34x - 130 \\
\hline
x - \frac{1}{2} & 2x^4 + x^3 - 35x^2 - 113x + 65 \\
\hline
& -2x^3 - x^2 \\
& \hline
& 2x^2 - 34x \\
& -2x^3 - x^2 \\
& \hline
& -34x + 13x \\
& -34x + 13x \\
& \hline
& -130x + 130 \\
& -130x + 65 \\
\end{array}
\]

(continued)
We now define this quotient as \( g(x) = 2x^3 + 2x^2 - 34x - 130 = 2(x^3 + x^2 - 17x - 65) \).
We continue with the list of zeroes candidates, beginning where we left off, in case \( x = \frac{1}{2} \) has a multiplicity greater than one:
\[
\begin{align*}
g \left( \frac{1}{2} \right) &= 2 \left[ \left( \frac{1}{2} \right)^3 + \left( \frac{1}{2} \right)^2 - 17 \left( \frac{1}{2} \right) - 65 \right] \neq 0, \\
g \left( -\frac{1}{2} \right) &= 2 \left[ \left( -\frac{1}{2} \right)^3 + \left( \frac{1}{2} \right)^2 - 17 \left( -\frac{1}{2} \right) - 65 \right] \neq 0, \\
g (5) &= 2 \left[ 5^3 + 5^2 - 17 \cdot 5 - 65 \right] = 2(125 + 25 - 85 - 65) = 0.
\end{align*}
\]
So \( x = 5 \) is the other positive real zero. If we now divide out the factor \((x - 5)\), we will have a quadratic polynomial which we can search more easily for the two remaining zeroes:
\[
\begin{align*}
\frac{x^2 + 6x + 13}{x-5} &\equiv \frac{x^3 + x^2 - 17x - 65}{x^2 - 5x} \\
&\equiv \frac{6x^2 - 17x}{6x^2 - 30x} \\
&\equiv \frac{13x - 65}{13x - 65}.
\end{align*}
\]
The discriminant of this quadratic polynomial is \( b^2 - 4ac = 6^2 - 4 \cdot 1 \cdot 13 = 36 - 52 < 0 \), so the two remaining zeroes form a complex conjugate pair (which also confirms that the number of negative real zeroes is indeed zero). If we solve the quadratic equation
\[
x^2 + 6x + 13 = 0,
\]
we find
\[
x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-6 \pm \sqrt{36 - 4 \cdot 13}}{2} = -3 \pm 2i.
\]
We have identified the four zeroes of \( f(x) \): two positive real, two complex. The polynomial may thus be factored as
\[
f(x) = (x - \frac{1}{2}) (x - 5) (x - [-3 + 2i]) (x - [-3 - 2i])
\]
\[
= (x - \frac{1}{2}) (x - 5) (x + 3 - 2i) (x + 3 + 2i).
\]

(10)

a) We will first wish to express the complex numbers
\( z = 1 - i \) and \( w = 1 - i \sqrt{3} \) in polar coordinates.

Since a complex number can be written as \( z = a + bi \), where \( |z| = \sqrt{a^2 + b^2} \), we find
\[
|z| = \sqrt{1^2 + (-1)^2} = \sqrt{2} \quad \text{and} \quad |w| = \sqrt{1^2 + (-\sqrt{3})^2} = 2.
\]
So
\[
z = \sqrt{2} \left( \frac{\sqrt{2}}{2} - i \frac{\sqrt{2}}{2} \right) \Rightarrow \cos \theta = \frac{\sqrt{2}}{2} \Rightarrow \theta = \frac{\pi}{4},
\]
\[
w = 2 \left( \frac{1}{2} - i \frac{\sqrt{3}}{2} \right) \Rightarrow \cos \theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}.
\]
(continued)
The multiplication rule for complex numbers gives

\[ z \cdot w = |z| \cdot |w| \cdot [\cos (\theta_1 + \theta_2) + i \sin (\theta_1 + \theta_2)] \]
\[ = \sqrt{2} \cdot 2 \cdot \left[ \cos \left( \frac{7\pi}{4} + \frac{5\pi}{3} \right) + i \sin \left( \frac{7\pi}{4} + \frac{5\pi}{3} \right) \right] \]
\[ = 2\sqrt{2} \left[ \cos \left( \frac{21\pi + 20\pi}{12} \right) + i \sin \left( \frac{21\pi + 20\pi}{12} \right) \right] \]
\[ = 2\sqrt{2} \left[ \cos \left( \frac{41\pi}{12} \right) + i \sin \left( \frac{41\pi}{12} \right) \right] = 2\sqrt{2} \left[ \cos \left( \frac{17\pi}{12} \right) + i \sin \left( \frac{17\pi}{12} \right) \right], \]

since polar angles for complex numbers are generally restricted to the principal circle \([0, 2\pi)\).

(we have applied \(\frac{41\pi}{12} - 2\pi = \frac{41\pi - 24\pi}{12} = \frac{17\pi}{12}\)).

The division rule for complex numbers leads to

\[ \frac{z}{w} = \frac{|z|}{|w|} \cdot \left[ \cos (\theta_1 - \theta_2) + i \sin (\theta_1 - \theta_2) \right] \]
\[ = \frac{\sqrt{2}}{2} \cdot \left[ \cos \left( \frac{7\pi}{4} - \frac{5\pi}{3} \right) + i \sin \left( \frac{7\pi}{4} - \frac{5\pi}{3} \right) \right] \]
\[ = \frac{\sqrt{2}}{2} \cdot \left[ \cos \left( \frac{21\pi - 20\pi}{12} \right) + i \sin \left( \frac{21\pi - 20\pi}{12} \right) \right] \]
\[ = \frac{\sqrt{2}}{2} \cdot \left[ \cos \left( \frac{\pi}{12} \right) + i \sin \left( \frac{\pi}{12} \right) \right]. \]

b) For the complex number \(16 - 16i\),

\[ |16 - 16i| = \sqrt{16^2 + (-16)^2} = \sqrt{256 + 256} \]
\[ = \sqrt{512} = 16\sqrt{2} \text{ or } 2^4 \cdot 2^{\frac{1}{2}} = 2^{\frac{9}{2}} \]
\[ \Rightarrow 16 - 16i = 16\sqrt{2} \left( \frac{\sqrt{2}}{2} - i \frac{\sqrt{2}}{2} \right) \]
\[ \Rightarrow \cos \theta = \frac{\sqrt{2}}{2} \quad \sin \theta = -\frac{\sqrt{2}}{2} \quad \Rightarrow \theta = \frac{7\pi}{4} \]
\[ \Rightarrow 16 - 16i = 2^{\frac{9}{2}} \left[ \cos \left( \frac{7\pi}{4} \right) + i \sin \left( \frac{7\pi}{4} \right) \right] \]

DeMoivre's Theorem for roots of complex numbers gives

\[ \sqrt[n]{z} = |z|^{\frac{1}{n}} \left[ \cos \left( \frac{\theta + 2k\pi}{n} \right) + i \sin \left( \frac{\theta + 2k\pi}{n} \right) \right], \quad k = 0, 1, 2, \ldots, (n-1). \]

The fourth roots of \(16 - 16i\) are thus:

\[ k = 0: \quad w_0 = \left[ 2^{\frac{9}{4}} \right]^{\frac{1}{4}} \left[ \cos \left( \frac{7\pi}{4} + 0 \right) + i \sin \left( \frac{7\pi+0}{4} \right) \right] = 2^{\frac{3}{2}} \left[ \cos \left( \frac{7\pi}{16} \right) + i \sin \left( \frac{7\pi}{16} \right) \right], \]
\[ k = 1: \quad w_1 = 2^{\frac{3}{2}} \cdot \left[ \cos \left( \frac{7\pi}{4} + 2\cdot1\cdot\pi \right) + i \sin \left( \frac{7\pi+2\cdot1\cdot\pi}{4} \right) \right] = 2^{\frac{3}{2}} \cdot \left[ \cos \left( \frac{15\pi}{16} \right) + i \sin \left( \frac{15\pi}{16} \right) \right], \]
\[ k = 2: \quad w_2 = 2^{\frac{3}{2}} \cdot \left[ \cos \left( \frac{7\pi}{4} + 2\cdot2\cdot\pi \right) + i \sin \left( \frac{7\pi+2\cdot2\cdot\pi}{4} \right) \right] = 2^{\frac{3}{2}} \cdot \left[ \cos \left( \frac{23\pi}{16} \right) + i \sin \left( \frac{23\pi}{16} \right) \right], \text{ and} \]
\[ k = 3: \quad w_3 = 2^{\frac{3}{2}} \cdot \left[ \cos \left( \frac{7\pi}{4} + 2\cdot3\cdot\pi \right) + i \sin \left( \frac{7\pi+2\cdot3\cdot\pi}{4} \right) \right] = 2^{\frac{3}{2}} \cdot \left[ \cos \left( \frac{31\pi}{16} \right) + i \sin \left( \frac{31\pi}{16} \right) \right]. \]
11. To find the equation for the parabola, the first aspect that will be of help is to recognize the orientation of the curve. The equation of the directrix, \( y = -2 \), is a horizontal line on a graph, which tells us that the symmetric axis of the parabola is vertical. So the equation for the parabola will have the form \( \pm 4p y = x^2 \). The focus lies at \((-4, 4)\), which is above the directrix on the graph, so the parabola "opens upward" and the coefficient of the \( y \)-term will be positive. The vertex and focus of the parabola are both on the symmetry axis; the vertex is midway between the focus and the directrix, so it is found at \((-4, \frac{4+(-2)}{2}) = (-4, 1)\). The distance from the vertex to either the focus or directrix is then \( p = |1-4| = |1-(-2)| = 3 \). As the vertex of our parabola is not at the origin, we must shift the parabola horizontally to the left by 4 units and vertically upward by 1 unit.

The equation for our parabola is thus given by \( +4p (y-k) = (x-h)^2 \)  
\[ 4 \cdot 3 \cdot (y-1) = (x-(-4))^2 \Rightarrow 12(y-1) = (x+4)^2. \] A graph is presented in the answer key.

12. We are given the equation of an ellipse in general form, \( x^2 + 2y^2 + 6x - 18y + 9 = 0 \); to extract the information in which we are interested, we must convert this to the standard form,  
\[ \frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1. \] In order to produce the needed binomial squares, we will need to use the "completing the square" method:

\[ (x^2 + 6x) + (y^2 - 18y) + 9 = 0 \]
\[ \Rightarrow (x^2 + 6x + 9) + 9(y^2 - 2y + 1) + 9 = 0 + 9 + 9 \]
\[ \Rightarrow (x+3)^2 + 9(y-1)^2 = 9 \]
\[ \Rightarrow \frac{(x+3)^2}{9} + \frac{(y-1)^2}{1} = 1. \]  

The center of the ellipse lies at \((h, k) = (-3, 1)\). Since the larger denominator is found with the "\( x^2 \)-term", the semi-major axis of length 3 is parallel to the \( x \)-axis; the semi-minor axis is 1 and is parallel to the \( y \)-axis. The vertices of the ellipse are at the ends of the major axis, so they are located at \((h \pm 3, k) = (-3 \pm 3, 1)\) or at \((-6, 1)\) and \((0, 1)\). The focal length of the ellipse is given by
\[ c^2 = a^2 - b^2 = 3^2 - 1^2 = 8 \Rightarrow c = \sqrt{8} = 2\sqrt{2}. \] The foci are thus located at \((h \pm c, k) = (-3 \pm 2\sqrt{2}, 1)\). A graph of this ellipse is shown in the answer key.
As in Problem 12, we wish to transform the equation of this hyperbola from the general form, 
\[2y^2 - x^2 + 2x + 8y + 3 = 0,\]
into standard form. Using again the "completing the square" technique, we find

\[(2y^2 + 8y) - x^2 + 2x + 3 = 0 \quad \text{grouping terms involving } x \ or \ y\]

\[\Rightarrow 2 \cdot (y^2 + 4y) - (x^2 - 2x) + 3 = 0 \quad \text{extracting common factors}\]

\[\Rightarrow 2 \cdot (y^2 + 4y + 4) - (x^2 - 2x + 1) + 3 = 0 + 2 \cdot 4 - 1\]

\[\Rightarrow 2 \cdot (y + 2)^2 - (x - 1)^2 + 3 = 7\]

\[\Rightarrow 2 \cdot (y + 2)^2 - (x - 1)^2 = 4\]

\[\Rightarrow \frac{(y + 2)^2}{2} - \frac{(x - 1)^2}{4} = 1 \quad \text{divide through by 4}\]

The center of the hyperbola lies at \((h, k) = (1, -2)\). The "y-term" is the positive term, so the symmetry axis of the branches of the hyperbola runs parallel to the y-axis. The "x-term" has the larger denominator, so the semi-major axis of length 2 is parallel to the x-axis, while the semi-minor axis has length \(\sqrt{2}\) and runs parallel to the y-axis, the symmetry axis. The vertices of the hyperbola are thus found at \((h, k \pm b) = (1, -2 \pm \sqrt{2})\). Since the focal length for a hyperbola is given by \(c^2 = a^2 + b^2 = 2^2 + (\sqrt{2})^2 = 6 \Rightarrow c = \sqrt{6}\), the foci of this hyperbola are located at \((h, k \pm c) = (1, -2 \pm \sqrt{6})\).

The asymptotes of the hyperbola form the diagonals of a box marked by the major and minor axes; the asymptotes cross each other at the center, \((1, -2)\). It can be seen that the slopes of these lines are \(\pm \frac{\sqrt{2}}{2}\), so the point-slope form of the equations for the asymptotes is

\[(y + 2) = \left(\frac{\sqrt{2}}{2}\right)(x - 1)\].

A graph of this hyperbola is shown in the answer key.
It is customary, in solving a set of linear equations by an elimination method, to write only the coefficients and constant terms in the equations. It is understood that the first column represents the x-terms; the second column, the y-terms; and so on. The row elimination process would go as follows:

\[
\begin{array}{ccc}
1 & 4 & -3 \\
3 & -1 & 3 \\
1 & 1 & 6
\end{array}
\rightarrow
\begin{array}{ccc}
1 & 4 & -3 \\
0 & -13/3 & 12 \\
0 & 3 & 9
\end{array}
\text{divide Row 2 by } -13
\]

\[
\begin{array}{ccc}
1 & 4 & -3 \\
0 & 1 & -12/3 \\
0 & 3 & 9
\end{array}
\rightarrow
\begin{array}{ccc}
1 & 4 & -3 \\
0 & 1 & -4 \\
0 & 3 & 9
\end{array}
\text{divide Row 3 by } 1/3
\]

\[
\begin{array}{ccc}
1 & 4 & -3 \\
0 & 1 & -4 \\
0 & 0 & 1
\end{array}
\rightarrow
\begin{array}{ccc}
1 & 0 & 3 \\
0 & 1 & -8/3 \\
0 & 0 & 1
\end{array}
\text{add 3 times Row 1}
\]

The solution set for our system of equations is \(x = 3, y = -\frac{8}{3}, z = \frac{1}{3}\). We can check these results in the original equations:

\[
3 + 4(-\frac{3}{2}) - 3(\frac{1}{3}) = 3 - \frac{21}{3} - \frac{1}{3} = -8; \quad 3(3) - (-\frac{3}{2}) + 3(\frac{1}{3}) = 9 - \frac{8}{3} + 1 = 12; \\
3 + (-\frac{3}{2}) + 6(\frac{1}{3}) = 3 - \frac{2}{3} + \frac{2}{3} = 2
\]

Since any sort of function could be in systems of non-linear equations, there is no general method for solving them; one simply has to be alert to possibilities for substitution or cancellation among the terms.

For the pair of equations, \(2x^2 - 3y^2 + 1 = 0\) and \(6x^2 - 7y^2 + 2 = 0\), it would be easiest to multiply the first equation by 3 and subtract it from the second one:

\[
2x^2 - 3y^2 + 1 = 0 \\
\text{subtract } 3 \text{ times Equation 1} \rightarrow 6x^2 - 7y^2 + 2 = 0 \Rightarrow \frac{2}{3}x^2 - \frac{1}{2}y^2 = 1 \Rightarrow \frac{3}{2}x^2 = 1 \Rightarrow x = \pm \sqrt{2}
\]

We can now substitute this result for \(y^2\) into the first equation to find

\[
\frac{2}{3}x^2 - \frac{1}{2} + \frac{1}{2} = 0 \Rightarrow \frac{3}{2}x^2 = \frac{1}{2} \Rightarrow x^2 = 4 \Rightarrow x = \pm 2.
\]

Every possible combination of these values for \(x\) and \(y\) is a solution to the pair of non-linear equations:

\((x, y) = (-2, -\sqrt{2}), (-2, \sqrt{2}), (2, -\sqrt{2}), (2, \sqrt{2})\).
In dealing with systems of inequalities with two variables, the solution will be a portion of a plane. It can be useful to consider the geometry of each curve described.

In our system, the first inequality is \( x - 2y \leq 6 \). If we solve this for \( y \), we obtain \( x - 2y \leq 6 \Rightarrow x - 6 \leq 2y \Rightarrow \frac{1}{2}x - 3 \leq y \). The equation \( y = \frac{1}{2}x - 3 \) describes a straight line, so the inequality represents the portion of the plane on or above it on a graph.

The second inequality, \( 2x + y \geq 2 \), leads to \( y \geq 2 - 2x \). The equation \( y = 2 - 2x \) also represents a straight line, so this inequality describes the portion of the plane on or above this line.

The final inequality, \( x^2 + y^2 \leq 36 \), need not be rearranged. The equation \( x^2 + y^2 = 36 \) describes a circle of radius 6 centered on the origin. Since the origin, \((0,0)\), satisfies the inequality \( x^2 + y^2 < 36 \), the given inequality describes the part of the plane on or inside the circle.

The solution set of this system of inequalities is the set of points that belong to all three of the sections described. A graph of this set is provided in the answer key.

An arithmetic series is a sum in which the first term is \( a \), the second term is \( a + d \), the third is \( a + 2d \), ..., and the \( n^{th} \) term is \( a + (n-1)d \), where \( d \) is the constant difference between each term. The sum of these \( n \) terms is

\[
S = a + (a+d) + (a+2d) + \ldots + (a+[n-1]d) = n \cdot a + \left( \frac{n \cdot [n-1]}{2} \right) \cdot d
\]

For our series, we are told that \( n = 19 \) and that

\[
S = 19 \cdot a + \left( \frac{19 \cdot 18}{2} \right) \cdot d = 19 \cdot a + \frac{19 \cdot 18}{2} \cdot (18 \cdot d) = 361.
\]

The final term is \( a + (19-1)d = a + 18d = 46 \). If we solve this equation for \( 18d = 46 - a \), we can immediately substitute this into the equation for the sum to find

\[
S = 19 \cdot a + \frac{19}{2} (46 - a) = 361
\]

\[
\Rightarrow 19a + 437 - \frac{19}{2}a = 361 \Rightarrow \frac{19}{2}a = -76 \Rightarrow a = -8.
\]

We can use this in the equation for the last term to obtain \( a + 18d = (-8) + 18 \cdot d = 46 \)

\[
\Rightarrow 18d = 54 \Rightarrow d = 3.
\]

Our arithmetic series is thus \((-8) + (-5) + (-2) + \ldots + 40 + 43 + 46 = 361\).
A geometric series is a sum in which the first term is \(a\), the second term is \(a \cdot r\), the third term is \(a \cdot r^2\), ..., and the \(n\)th term is \(a \cdot r^{n-1}\), where \(r\) is the constant ratio between each term. An infinite geometric series

\[
S = a + a \cdot r + a \cdot r^2 + \ldots + a \cdot r^n + \ldots = \sum_{k=0}^{\infty} a \cdot r^k
\]

only has a meaningful sum if \(|r| < 1\); that sum is \(S = \frac{a}{1-r}\).

When applying this formula, it is important to verify what the first term is.

For our series, \(\sum_{k=1}^{\infty} 3 \cdot \left(-\frac{3}{4}\right)^k\) written out would be

\[
3 \cdot \left(-\frac{3}{4}\right)^1 + 3 \cdot \left(-\frac{3}{4}\right)^2 + 3 \cdot \left(-\frac{3}{4}\right)^3 + \ldots = -\frac{9}{4} + \frac{27}{16} - \frac{81}{64} + \ldots
\]

So the initial term is \(a = -\frac{9}{4}\) and the constant ratio is \(r = -\frac{3}{4}\). Since \(|r| = \frac{3}{4} < 1\), this series has the infinite sum \(S = \frac{-\frac{9}{4}}{1 - (-\frac{3}{4})} = \frac{-\frac{9}{4}}{\frac{7}{4}} = -\frac{9}{7}\).

G. Ruffa -- March 2007