1) In order to find the value of $\tan \theta$, we will need to know the value of $\cos \theta$. We are given a value for $\sin \theta$ which is positive, and told that $\cos \theta$ is negative, which means that $\theta$ is in the fourth quadrant. We will not need to know the actual value for the angle $\theta$, however, since we can find the cosine from the Pythagorean Identity:

$$\sin^2 \theta + \cos^2 \theta = 1 \implies \cos^2 \theta = 1 - \sin^2 \theta$$

$$\implies \cos \theta = \pm \sqrt{1 - \sin^2 \theta} = \pm \sqrt{1 - \left(\frac{\sqrt{3}}{2}\right)^2} = \pm \sqrt{1 - \frac{3}{4}} = \pm \frac{1}{2} .$$

We will want the negative square root here, so the value of the tangent function is

$$\tan \theta = \frac{\sin \theta}{\cos \theta} = \frac{\sqrt{3}/2}{-1/2} = -\sqrt{3} . \quad (B)$$

2) This problem is more of an exercise in recognition of useful trigonometric identities. The basic form of the “double-angle” identity for cosine is $\cos 2\theta = \cos^2 \theta - \sin^2 \theta$. If we combine this with the Pythagorean Identity, $\sin^2 \theta + \cos^2 \theta = 1$, we can write two “alternative” forms of this expression:

$$\cos 2\theta = \cos^2 \theta - \sin^2 \theta = (1 - \sin^2 \theta) - \sin^2 \theta = 1 - 2 \sin^2 \theta \quad \text{or}$$

$$\cos 2\theta = \cos^2 \theta - \sin^2 \theta = \cos^2 \theta - (1 - \cos^2 \theta) = 2 \cos^2 \theta - 1 .$$

We see from this that the only choice containing an incorrect expression is $\text{(C)}$.

3) For this calculation, we will use DeMoivre’s Theorem for powers of complex numbers. In order to apply it, however, we must first express the complex number $z$ in polar form. The complex number $z = a + bi$ is thus written as $z = r \cdot (\cos \theta + i \sin \theta)$, with

$$r = \sqrt{a^2 + b^2} = \sqrt{\left(-\frac{\sqrt{2}}{2}\right)^2 + \left(\frac{\sqrt{2}}{2}\right)^2} = \sqrt{\frac{1}{2} + \frac{1}{2}} = 1 ,$$

$$\theta = \tan^{-1}\left(\frac{b}{a}\right) \quad \text{[appropriate quadrant]} = \tan^{-1}\left(\frac{\sqrt{2}/2}{-\sqrt{2}/2}\right) \quad \text{[quadrant II]}$$

$$= \tan^{-1}(-1) \quad \text{[quadrant II]} = \frac{3\pi}{4} .$$

So our complex number can be written as $z = 1 \cdot (\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4}) . \quad \text{(continued)}$
DeMoivre’s Theorem tells us that $w = z^n = r^n \cdot (\cos[n\theta] + i \sin[n\theta])$, so we have

$$w = z^5 = 1^5 \cdot (\cos[5 \cdot \frac{3\pi}{4}] + i \sin[5 \cdot \frac{3\pi}{4}]) = 1 \cdot (\cos\frac{15\pi}{4} + i \sin\frac{15\pi}{4})$$

$$= \cos \frac{7\pi}{4} + i \sin \frac{7\pi}{4} = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} \cdot i \quad \text{(A)}$$

4) The equation for the ellipse is written in the standard form for a conic section, but this is not the most convenient form from which to obtain information about the curve. We can “complete the squares” and write terms as “binomial-squares” in order to express this as the equation for an ellipse:

$$2x^2 + 3y^2 - 4x + 6y = 0 \Rightarrow 2x^2 - 4x + 3y^2 + 6y = 0 \Rightarrow 2(x^2 - 2x) + 3(y^2 + 2y) = 0$$

$$\Rightarrow 2(x^2 - 2x + 1) + 3(y^2 + 2y + 1) = 0 + 2 \cdot 1 + 3 \cdot 1 \Rightarrow 2(x - 1)^2 + 3(y + 1)^2 = 5$$

$$\Rightarrow \frac{2(x - 1)^2}{5} + \frac{3(y + 1)^2}{5} = 1 \Rightarrow \frac{(x - 1)^2}{\frac{5}{2}} + \frac{(y + 1)^2}{\frac{5}{3}} = 1 \quad \text{(E)}$$

From this, we can extract the information that the center of this ellipse lies at $(1, -1)$.

5) The equations for transforming the polar coordinates $(r, \theta)$ of a point into rectangular (Cartesian) coordinates $(x, y)$ are $x = r \cos \theta, y = r \sin \theta$. An additional adjustment must be made for the point in question in this problem, since the radius $r$ is given as negative: in such a case, we use the relation $(−r, \theta) = (r, \theta \pm \pi)$.

For our point, then, we can write

$$(-2, -\frac{\pi}{3}) = (2, -\frac{\pi}{3} + \pi) = (2, \frac{2\pi}{3})$$

$$\Rightarrow (2 \cos \frac{2\pi}{3}, 2 \sin \frac{2\pi}{3}) = (2 \cdot [−\frac{1}{2}], 2 \cdot \frac{\sqrt{3}}{2}) = (-1, \sqrt{3}) \quad \text{(E)}$$

6) There is no general method for solving a system of non-linear equations, since the approach to take depends on the form of the equations. Instead, we need to look at the relationships in the given equations that may suggest a way to solve them. For the pair of equations given, we notice that both have a linear term involving $y$ -- this allows us to make a substitution of the first equation into the second one:

$$-x + y = 1 \Rightarrow y = x + 1$$

$$x^2 + 4y = 1 \Rightarrow x^2 + 4 \cdot (x + 1) = 1 \Rightarrow x^2 + 4x + 4 = 1 \Rightarrow (x + 2)^2 = 1$$

$$\Rightarrow x + 2 = \pm 1 \Rightarrow x = -3, x = -1 \quad *$$

* This result alone already eliminates all but one of the choices, but we will go on to check the values for $y$.

(continued)
We now insert these values into either of the original equations to find the corresponding values of $y$:

$$x = -3 \Rightarrow y = x + 1 = (-3) + 1 = -2 \quad ; \quad x = -1 \Rightarrow y = (-1) + 1 = 0 .$$

The ordered pair solutions for our system of equations is thus $(-3, -2)$ and $(-1, 0)$.

These ordered pairs can also be seen as providing the coordinates of the two points of intersection between the line $y = x + 1$ and the horizontal parabola $4y = 1 - x^2$.

7) Since the general term of this series involves a power of a constant value, this is a geometric series. The factor raised to a power is the ratio between successive terms, $r = \frac{1}{2}$. The sum is taken from index $k = 0$ to $k = 40$, so the initial term of the series is $a = (\frac{1}{2})^0 = 1$ and the number of terms in the series is $n = 41$. We can then use the formula for the sum of a finite geometric series to find

$$s_n = a \cdot \frac{1-r^n}{1-r} \Rightarrow s_{41} = 1 \cdot \frac{1-(\frac{1}{2})^{41}}{1-(\frac{1}{2})} = 2 \left[1-(\frac{1}{2})^{41}\right] = 2 - 2 \cdot (\frac{1}{2})^{41} = 2 - (\frac{1}{2})^{40} .$$

8) Since this is a fourth-degree polynomial, it is not generally easy to simply spot possible zeroes. Use of the Rational Zeroes Theorem suggests itself in such a situation: we would find candidates for zeroes by dividing all possible factors of the constant term by all possible factors of the leading coefficient. Unfortunately, here the constant term is 6 and the leading coefficient is 1, so the candidate zeroes are ±1, ±2, ±3, and ±6, which eliminates absolutely none of the choices offered.

The only thing for it, then, is to test the different choices for $x$ in the polynomial. We can see relatively quickly that

$$x = 1 : 1^4 - 4 \cdot 1^3 + 5 \cdot 1 + 6 = 1 - 4 + 5 + 6 \neq 0 ,$$
$$x = -1 : (-1)^4 - 4 \cdot (-1)^3 + 5 \cdot (-1) + 6 = 1 + 4 - 5 - 6 \neq 0 .$$

Upon further testing, we find that $2^4 - 4 \cdot 2^3 + 5 \cdot 2 + 6 = 16 - 32 + 10 + 6 = 0$, so $x = 2$ is a zero of our polynomial.

9) The center, vertices, and foci of an ellipse all lie along its major axis; since the given points all have the $x$-coordinate of 0, the major axis must lie on the $y$-axis. The center of the ellipse is midway between the two vertices, so it is located at $(0, \frac{4+(-4)}{2}) = (0, 0)$, in other words, at the origin. The semi-major axis is the distance from the center of the ellipse to either vertex, so $a = 4$. The focal distance $c$ is the distance from the center to either focus; we are told that one focus lies at $(0, -2)$, so $c = 2$. This permits us to calculate the semi-minor axis, $b$, from the relation $a^2 - b^2 = c^2 \Rightarrow a^2 - c^2 = b^2 \Rightarrow 4^2 - 2^2 = 16 - 4 = 12 = b^2$. 

(continued)
This now provides us with sufficient information to write the equation for the ellipse. Since the major axis is on the \( y \)-axis (that is to say, it is vertical), the value for \( a^2 \) will go with the \( y^2 \)-term in the equation

\[
\left( \frac{x - h}{b} \right)^2 + \left( \frac{y - k}{a} \right)^2 = 1 \quad \Rightarrow \quad \frac{(x - 0)^2}{12} + \frac{(y - 0)^2}{4^2} = 1 \quad \Rightarrow \quad \frac{x^2}{12} + \frac{y^2}{16} = 1 .
\]  

\( \text{(A)} \)

10) We can work out this problem similarly to the way we did Problem 1. We have

\[
\cos \theta = \pm \sqrt{1 - \sin^2 \theta} = \pm \sqrt{1 - \left( \frac{3}{5} \right)^2} = \pm \sqrt{1 - \frac{9}{25}} = \pm \frac{\sqrt{16}}{5} = \pm \frac{4}{5} .
\]

We are told that the angle \( \theta \) is in the first quadrant, so we take the positive value for \( \cos \theta \). The value for \( \sin 2\theta \) is then

\[
\sin 2\theta = 2 \sin \theta \cos \theta = 2 \cdot \frac{3}{5} \cdot \frac{4}{5} = \frac{24}{25} .
\]  

\( \text{(D)} \)

11) We can use Descartes' Rule of Signs to get an idea of the number of positive or negative real zeroes of a polynomial. The number of sign changes among the terms for \( f(x) \) tells us whether the number of positive real zeroes is even or odd, with the maximum number being the number of sign changes; the number of sign changes among the terms for \( f(-x) \) gives us similar information for the negative real zeroes.

For our polynomial, we see that

\[
f(x) = -4x^6 + 7x^5 - 3x^4 + 5x^3 - 8x^2 - x + 2 ,
\]

\[\wedge \quad \wedge \quad \wedge \quad \wedge \quad \wedge \]

5 sign changes

which tells us that there are an odd number of positive real zeroes, the number being five, three, or one. (It turns out there is just one.) We can obtain the expression for \( f(-x) \) by simply changing the signs on the coefficients of the terms with odd powers of \( x \), giving us

\[
f(-x) = -4x^6 - 7x^5 - 3x^4 - 5x^3 - 8x^2 + x + 2 ,
\]

\[\wedge \quad \wedge \quad \wedge \quad \wedge \quad \wedge \]

1 sign change

so the number of negative real zeroes is odd and no larger than one, so there must be only one negative real zero.  

\( \text{(B)} \)

12) In working with systems of linear inequalities, it is helpful to work with the equation parts first and write the equations for the lines in slope-intercept form:

\[
x - y + 2 = 0 \quad \Rightarrow \quad y = x + 2 , \quad 2y + 3x - 2 = 0 \quad \Rightarrow \quad y = -\frac{3}{2}x + 1 .
\]

This allows us to plot the two lines that act as boundaries for the region that is the solution to the pair of inequalities (see the graph presented in the Answer Key).  

(continued)
We can now look at the inequalities themselves. The first inequality can be re-arranged to read  \( x - y + 2 \geq 0 \Rightarrow x + 2 \geq y \), which means that the \( y \)-coordinate of points in this region are less than or equal to the values on the line \( y = x + 2 \), so those points lie on or “below” the line (the red line and the region shown in pale red or violet). In the same manner, we can re-arrange the second inequality to read  \( 2y + 3x - 2 \leq 0 \Rightarrow y \leq -\frac{3}{2}x + 1 \), which tells us that points described by this inequality lie on or “below” the line \( y = -\frac{3}{2}x + 1 \) (the blue line and the region shown in light blue or violet). To satisfy both inequalities at the same time, a point must lie in the section common to both of these regions, which is the “overlap” shown in violet, along with the portions of the red and blue lines on its edges.

13) To solve a trigonometric equation, it is necessary that all of the terms have the same argument in any trigonometric functions which appear. So, to start from the equation  \( \cos \theta + \sin 2\theta = 0 \), we will need to express all of the terms using \( \theta \) only.

By using the “double-angle” identity for sine, we can write

\[
\cos \theta + \sin 2\theta = 0 \Rightarrow \cos \theta + 2 \sin \theta \cos \theta = 0 \Rightarrow \cos \theta \cdot (1 + 2 \sin \theta) = 0 .
\]

The product of factors is zero if any of the factors is zero. Since this problem asks for all possible solutions, we will solve for the solutions on the fundamental circle, \( [0, 2\pi) \), and then add the term \( 2k\pi \), \( k \) being any integer, to cover the infinite number of solutions. Hence the set of all solutions to our equation is

\[
\cos \theta = 0 \Rightarrow \theta = \frac{\pi}{2} + 2k\pi , \quad \frac{3\pi}{2} + 2k\pi \quad \text{or} \quad \frac{\pi}{2} + k\pi ;
\]

\[
1 + 2 \sin \theta = 0 \Rightarrow \sin \theta = -\frac{1}{2} \Rightarrow \theta = \frac{7\pi}{6} + 2k\pi , \quad \frac{11\pi}{6} + 2k\pi .
\]

14) To work out the exact value for the trigonometric function of a specific angle, we need to be able to express that angle in a simple way compared to angles for which we already know exact values of trigonometric functions. We can then use the identities with which we are familiar to work out a value for the given angle.

In the case of \( \theta = \frac{9\pi}{8} \), we want to see first whether it can be easily written as a sum or difference of two angles, as this would allow us to calculate \( \sin \left( \frac{9\pi}{8} \right) \) using the “angle-addition” formulas. Unfortunately, the angles for which we know trigonometric values are multiples of \( \frac{\pi}{6} \) or \( \frac{\pi}{4} \); since \( \frac{9\pi}{8} = \frac{27\pi}{24} = \frac{4\pi}{24} \), and \( \frac{\pi}{4} = \frac{6\pi}{24} \), there is no convenient sum or difference we can write to express the angle of concern to us.

Since half of the angle is \( \frac{9\pi}{16} \), for which we also don’t know any trigonometric values, we won’t be able to use the “double-angle” formulas. This leaves us only the option of noting that \( \frac{9\pi}{8} = \frac{1}{2} \cdot \frac{9\pi}{4} \), which will permit us to use the “half-angle” formulas.

(continued)
If we start from the “double-angle” formula for cosine, together with the Pythagorean Identity, we obtain

\[
\cos 2\theta = \cos^2 \theta - \sin^2 \theta = (1 - \sin^2 \theta) - \sin^2 \theta = 1 - 2\sin^2 \theta
\]

\[
\Rightarrow 2\sin^2 \theta = 1 - \cos 2\theta \quad \Rightarrow \quad \sin^2 \theta = \frac{1 - \cos 2\theta}{2} \quad \Rightarrow \quad \sin \theta = \pm \sqrt{\frac{1 - \cos 2\theta}{2}};
\]

keeping in mind that \(\theta = \frac{1}{2} \cdot 2\theta\), we can then write a “half-angle” formula for sine,

\[
\sin \left(\frac{\theta}{2}\right) = \pm \sqrt{\frac{1 - \cos \theta}{2}},
\]

with the sign of the square root chosen appropriately for the quadrant in which \(\theta/2\) is found. Upon noting that \(\frac{9\pi}{8}\) lies in the third quadrant, we are now in a position to calculate

\[
\sin \left(\frac{9\pi}{8}\right) = -\sqrt{\frac{1 - \cos \frac{9\pi}{4}}{2}} = -\sqrt{\frac{1 - \cos \left(\frac{\pi}{4} + \frac{2\pi}{4}\right)}{2}} = -\sqrt{\frac{1 - \cos \frac{\pi}{4}}{2}}
\]

\[
= -\sqrt{\frac{1 - \frac{\sqrt{2}}{2}}{2} \cdot \frac{2}{2}} = -\sqrt{\frac{2 - \sqrt{2}}{4}} = -\sqrt{\frac{2 - \sqrt{2}}{2}} \approx -0.3827.
\]

It can be confirmed using a calculator that \(\sin 202.5^\circ \approx -0.3827\).

15) To begin solving this triangle, we note that we are given the lengths of two sides and the measure of the angle between them (the “included angle”). This allows us to use the Law of Cosines to find

\[
c^2 = a^2 + b^2 - 2ab \cos \gamma = 5^2 + 10^2 - 2 \cdot 5 \cdot 10 \cos 75.5^\circ = 25 + 100 - 100 \cdot 0.2504 = 99.962.
\]

The three sides of our triangle are thus \(a = 5, b = 10, c = 9.998\). (It was likely intended by the test-maker that \(c\) come out to be exactly 10, making this an isosceles triangle.)

We can now apply the Law of Sines to find all of the angles in this triangle:

\[
\frac{\sin \alpha}{a} = \frac{\sin \beta}{b} = \frac{\sin \gamma}{c} \quad \Rightarrow \quad \frac{\sin \alpha}{5} = \frac{\sin \beta}{10} = \frac{\sin 75.5^\circ}{9.998}
\]

\[
\Rightarrow \quad \sin \alpha = \frac{5}{9.998} \cdot \sin 75.5^\circ = 0.5001 \cdot 0.9681 = 0.48415 \quad \Rightarrow \quad \alpha = 28.96^\circ,
\]

\[
\sin \beta = \frac{10}{9.998} \cdot \sin 75.5^\circ = 1.0002 \cdot 0.9681 = 0.96839 \quad \Rightarrow \quad \beta = 75.56^\circ.
\]

This may, of course, be fussier than we needed to be. Since sides \(b\) and \(c\) are practically the same length, we have an isosceles triangle, so \(\beta = \gamma = 75.5^\circ\). In that case, the remaining angle simply makes up the rest of the 180\(^\circ\) total for the three angles of a triangle, hence \(\alpha = 180^\circ - \beta - \gamma = 180^\circ - 2\gamma = 180^\circ - (2 \cdot 75.5^\circ) = 29.0^\circ\).
This identity can be proven starting from either side of the equation, but it is more convenient to begin with the ratio on the right-hand side:

\[
\frac{\cos 2\theta}{\sin^2 2\theta} = \frac{\cos 2\theta}{(\sin 2\theta)^2} = \frac{\cos^2 \theta - \sin^2 \theta}{(2 \sin \theta \cos \theta)^2} = \frac{\cos^2 \theta}{4 \sin^2 \theta \cos^2 \theta} - \frac{\sin^2 \theta}{4 \sin^2 \theta \cos^2 \theta}
\]

\[
= \frac{1}{4 \sin^2 \theta} - \frac{1}{4 \cos^2 \theta} = \frac{1}{4} \csc^2 \theta - \frac{1}{4} \sec^2 \theta,
\]

thereby establishing the identity.

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