We've mentioned that definite integration gives a signed or net area "below" the curve for \( f(x) \): area above the \( x \)-axis is counted as positive, but area below the \( x \)-axis is reckoned as negative, owing to the Riemann sum being \( \lim_{n \to \infty} \Delta x \cdot \sum_{i=1}^{n} f(x_i) \). For any quantity which can be given by a function \( F(x) \), with a known rate \( F'(x) = \frac{dF}{dx} \), the integral

\[
\int_{a}^{b} F(x) \, dx = F(b) - F(a)
\]

is the net change in the quantity \( F(x) \) over the interval from \( a \) to \( b \), \( \Delta F = F(b) - F(a) \), provided \( F(x) \) is continuous on \([a, b]\). This is the Net Change Theorem.

The net change in a quantity can be useful to know, but sometimes we need information which cannot be found by simply integrating a rate. We can look at Problem 58 in Section 5.4: we have an object which moves in accordance with the velocity function \( v(t) = t^2 - 2t - 8 \) during the interval \( 1 \leq t \leq 6 \). So, for part (a), the net change in position of the object (also called the (net) displacement) is found from

\[
\Delta x = \int_{1}^{6} (\frac{dx}{dt}) \, dt = \int_{1}^{6} v(t) \, dt = \int_{1}^{6} t^2 - 2t - 8 \, dt
\]

\[
= \left[ \frac{1}{3} t^3 - t^2 - 8t \right]_{1}^{6} = \left( \frac{1}{3} \cdot 6^3 - 6^2 - 8 \cdot 6 \right) - \left( \frac{1}{3} \cdot 1^3 - 1^2 - 8 \cdot 1 \right)
\]

\[
= \left( 216 - 36 - 48 \right) - \left( \frac{1}{3} - 1 - 8 \right) = -\frac{10}{3}
\]

This tells us that the object ends up at \( t = 6 \) at a position \( \frac{10}{3} \) units to the left of where it was at \( t = 1 \) (taking motion to the right as positive).
This, however, does not tell us anything about how the object traveled during the interval. We must investigate the velocity function more thoroughly to answer part (b), concerning the total distance traveled during $1 \leq t \leq 6$.

The velocity function can be factored as $v(t) = t^2 - 2t - 8 = (t+2)(t-4)$, so it is an upward-facing parabola with $x$-intercepts at $t=-2$ and $t=+4$. We find that $v(t) < 0$ for $1 \leq t < 4$ and $v(t) > 0$ for $4 < t \leq 6$; the object is moving "to the left" for these first three seconds, stops momentarily at $t=4$, and then moves off "to the right" during the last two seconds. The change in position during $1 \leq t < 4$ is counted as negative in computing the net displacement of the object. So, to find the total distance that the object has traveled, we need to take the absolute value of that area. This makes the integral

$$\int_1^4 |t^2 - 2t - 8| \, dt + \int_4^6 t^2 - 2t - 8 \, dt$$

This area is already positive

$$= \int_1^4 -(t^2 - 2t - 8) \, dt + \int_4^6 t^2 - 2t - 8 \, dt$$

$$= \left( -\frac{1}{3}t^3 + t^2 + 8t \right)_1^4 + \left( \frac{1}{3}t^3 - t^2 - 8t \right)_4^6$$

$$= \left[ (-\frac{64}{3} + 16 + 32) - (-\frac{1}{3} + 1 + 8) \right] + \left[ (\frac{81}{3} - 36 - 48) - (\frac{64}{3} - 16 - 32) \right]$$

$$= \left( \frac{54}{3} \right) + \left( \frac{44}{3} \right) = \frac{98}{3}.$$

So the history of the particle's motion can be rendered as
Now we know how to work out a few anti-derivatives, so that we're able to write some indefinite integrals and use that, through the aid of the Fundamental Theorem of (Integral) Calculus, to compute definite integrals. We're still pretty seriously limited, though; we only know how to find anti-derivatives if we happen to know a function having that particular derivative function. So we can figure out
\[ \int e^x \, dx \quad \text{or} \quad \int \cos x \, dx , \]
but what about \[ \int e^{3x} \, dx \quad \text{or} \quad \int \cos \left( \frac{x}{2} - \frac{\pi}{6} \right) , \]
much less
\[ \int \frac{4x}{\sqrt{9x^2 - 1}} \, dx \quad \text{or} \quad \int \cos \theta \sin^6 \theta \, d\theta ? \]

We are now setting out to learn "techniques of integration", which will permit us to solve integrals when we can't just "spot" the answer from our knowledge of simple derivatives. We introduce the simplest of these methods in Section 5.5 and will see many others in Chapter 7.

We know that \[ \int e^x \, dx = e^x + C \] because \[ \frac{d}{dx} e^x = e^x \], so \( e^x \) manages to be its own derivative and anti-derivative. When we differentiate \( e^{3x} \), we notice that
\[ \frac{d}{dx} e^{3x} = \frac{d}{du} e^u \cdot \frac{du}{dx} = e^u \cdot \frac{d}{dx} (3x) = e^{3x} \cdot 3 . \]
This differentiation contributed a multiplicative factor of 3, so if we'd started with \( \frac{1}{3} \cdot e^{3x} \), we would get
\[ \frac{d}{dx} \left( \frac{1}{3} e^{3x} \right) = \frac{1}{3} \cdot e^{3x} \cdot 3 = \frac{1}{3} e^{3x} . \]
We see from this that \[ \int e^{3x} \, dx = \frac{1}{3} e^{3x} + C . \]
So let's try $\int \sin(\pi t) \, dt$: the anti-derivative of $\sin t$ is $-\cos t + C$, so if we differentiate $-\cos(\pi t) + C$, we would obtain

$$\frac{d}{dt} \left[ -\cos(\pi t) + C \right] = -\frac{d}{du} \cos u \cdot \frac{du}{dt} = -(-\sin u) \cdot \frac{d}{dt} (\pi t) = \sin (\pi t) \cdot \pi.$$

From this, we can see that if we had set up $-\frac{1}{\pi} \cos(\pi t) + C$, the derivative would be $-\frac{1}{\pi} \cdot (-\sin(\pi t)) \cdot \pi = \sin (\pi t)$. So $\int \sin (\pi t) \, dt = -\frac{1}{\pi} \cos(\pi t) + C$.

Is the method then just a matter of bringing in some constant? Look at $\int \sqrt{4x-7} \, dx$ — we don't even know an elementary function that has a derivative resembling $\sqrt{4x-7}$. But if we look at the derivative of this function, we notice that

$$\frac{d}{dx} \sqrt{4x-7} = \frac{d}{dx} \left( \frac{4x-7}{2} \right)^{\frac{1}{2}} = \frac{d}{du} u^{\frac{1}{2}} \cdot \frac{du}{dx} = \frac{1}{2} u^{-\frac{1}{2}} \cdot \frac{d}{dx} (4x-7) = \frac{1}{2} (4x-7)^{-\frac{1}{2}} \cdot 4 .$$

We know how to anti-differentiate $x^{\frac{1}{2}}$, so let's see what happens when we raise the exponent for $4x-7$ by 1:

$$\frac{d}{dx} (4x-7)^{\frac{1}{2}+1} = \frac{d}{dx} \left( \frac{4x-7}{2} \right)^{\frac{3}{2}} = \frac{3}{2} u^{\frac{1}{2}} \cdot \frac{d}{dx} (4x-7) = \frac{3}{2} (4x-7)^{\frac{3}{2}} \cdot 4 .$$

If we wanted to be sure to end up with just $\sqrt{4x-7}$, we would include a factor $\frac{1}{\frac{3}{2}} = \frac{2}{3}$ to cancel out the $\frac{3}{2}$ from differentiation (we already know this from $\int x^{\frac{1}{2}} \, dx = \frac{2}{3} x^{\frac{3}{2}} + C$) and a factor of $\frac{1}{4}$ to cancel out the $\frac{du}{dx} = 4$ from the Chain Rule. The indefinite integral is then

$$\int \sqrt{4x-7} \, dx = \frac{1}{\frac{3}{2}} \cdot \frac{1}{4} \cdot (4x-7)^{\frac{3}{2}} + C = \frac{1}{6} (4x-7)^{\frac{3}{2}} + C .$$
So perhaps we need a method that does for integration something like what the Chain Rule does for differentiation: find a part of the composite function to call "u", separate out of the integrand the portion corresponding to \( \frac{du}{dx} \) \( dx = du \), and turn the integral into one for which we do know an anti-derivative:

\[
\int f(u) \ du = \int \frac{f(u)}{\frac{du}{dx}} \ dx = \int f(u) \ \frac{du}{dx} = \int f(u) \ du = F(u) + C
\]

\[
\Rightarrow \ F(x) + C.
\]

For the anti-derivative we just worked out, this approach looks like

\[
\int \frac{\sqrt{4x-7}}{u} \ dx \quad \Rightarrow \quad u = 4x - 7
\]

\[
\Rightarrow \ \frac{du}{dx} = 4 \quad \Rightarrow \ du = 4 \ dx \quad \Rightarrow \ \frac{1}{4} \ du = dx
\]

we re-write this as a differential

\[
\Rightarrow \ \frac{1}{4} \int \sqrt{u} \ du = \frac{1}{4} \int u^{\frac{1}{2}} \ du = \frac{1}{2} \cdot \frac{1}{\frac{3}{2}} \cdot u^{\frac{3}{2}} + C = \frac{1}{\frac{3}{2}} \cdot u^{\frac{3}{2}} + C
\]

This, we know how to solve.

\[
= \frac{1}{6} \ u^{\frac{3}{2}} + C
\]

restoring \( u \)

in terms of \( x \)

check:

\[
\frac{d}{dx} \left[ \frac{1}{6} \ (4x-7)^{\frac{3}{2}} \right] = \frac{1}{6} \cdot \frac{3}{2} \ u^{\frac{1}{2}} \cdot \frac{du}{dx} = \frac{1}{6} \cdot \frac{3}{2} \cdot 4 \cdot u^{\frac{1}{2}}
\]

\[
= \frac{12}{12} \ (4x-7)^{\frac{3}{2}} = \sqrt{4x-7}.
\]

We can return to our earlier example with this "substitution method" now—we can even solve them generically!

\[
\int e^{kx} \ dx \quad u = kx \Rightarrow \ \frac{du}{dx} = k \Rightarrow \ du = k \ dx \Rightarrow \ \frac{1}{k} \ du = dx
\]

\[
\Rightarrow \ \int e^{u} \cdot \frac{1}{k} \ du = \frac{1}{k} \int e^{u} \ du = \frac{1}{k} \cdot e^{u} + C \quad \rightarrow \quad \frac{1}{k} e^{kx} + C;
\]

\[
\int \sin \frac{u}{k} \ dt \quad u = kt \Rightarrow \ du = k \ dt \Rightarrow \ dt = \frac{1}{k} \ du
\]

\[
\Rightarrow \ \int \sin u \cdot \frac{1}{k} \ du = \frac{1}{k} \int \sin u \ du = \frac{1}{k} \cdot (-\cos u) + C \quad \rightarrow \quad -\frac{1}{k} \cos kt + C
\]

Next time: tougher integrals