We are asked to find the points \((X, Y)\) on the ellipse \(4x^2 + y^2 = 4\) that are farthest away from the point \((1, 0)\); as it happens, this point is also on the ellipse, but that is actually of no importance to the solution. We can use the distance formula to calculate the separation of a hypothetical farthest point from the point \((1, 0)\) from \[s^2 = (X-1)^2 + (Y-0)^2.\] Since the point \((X, Y)\) is on the ellipse, we can write \[4X^2 + Y^2 = 4 \implies Y^2 = 4 - 4X^2.\] When we insert this into the distance equation, we obtain

\[s^2 = (X-1)^2 + Y^2 = (X-1)^2 + (4 - 4X^2) = X^2 - 2X + 1 + 4 - 4X^2 = 5 - 2X - 3X^2.\]

It is the distance \(s\) we wish to minimize, but since it is a positive number, minimizing \(s^2\) will serve as well. We can thus differentiate \(s^2\) with respect to \(X\) and set that derivative equal to zero, in order to locate critical numbers:

\[
\frac{ds^2}{dX} = \frac{d}{dX}(5 - 2X - 3X^2) = -2 - 6X = 0 \implies 6X = -2 \implies X = -\frac{1}{3}.
\]

We can now put this critical number into the equation for the ellipse in order to find the \(Y\)-coordinates of the “most distant” points:

\[Y^2 = 4 - 4X^2 = 4 - 4\cdot\left(-\frac{1}{3}\right)^2 = 4 - \frac{4}{9} = \frac{36 - 4}{9} = \frac{32}{9}.
\]

\[\implies Y = \pm\sqrt{\frac{32}{9}} = \pm\sqrt{\frac{2\cdot16}{9}} = \pm\frac{4}{3}\sqrt{2} = \pm1.886.\] (continued)
There are two symmetrically-placed “most distant” points at \((-\frac{1}{3}, \frac{4}{3} \sqrt{2})\) and \((-\frac{1}{3}, -\frac{4}{3} \sqrt{2})\), as we would expect from the geometry of the ellipse; what is perhaps surprising is that the farthest points are not at the ends of the long axis of the ellipse.

We see that the second derivative is \(\frac{d^2}{dX^2}(s^2) = -6 < 0\), so the critical point for the distance-squared (and thus the distance) is indeed a maximum.

27)

![Diagram of a right circular cylinder inscribed in a sphere.](image)

A right circular cylinder is one for which the cross-sections are circles and the axis through the centers of those circles is perpendicular to the base (in fact, what most people mean by a “cylinder”). Having this cylinder “inscribed” within a sphere means that the edges of the top and bottom ends of the cylinder are in contact with the inside of the sphere. We wish to find the cylinder with the largest volume that can fit inside a sphere of radius \(r\).

If we were to chop through the cylinder and the sphere along the symmetry axis of the cylinder, we would see a cross-section with a rectangle touching a circle at each of the rectangle’s corners. We can place one of these corners at \((x, y)\), which would put the other corners at \((x, -y)\), \((-x, y)\), and \((-x, -y)\). This would give our inscribed cylinder a base radius of \(x\) and a height of \(2y\), making its volume \(V = \pi x^2 \cdot (2y) = 2\pi x^2 y\); this is the function we wish to maximize. We do know one other thing: since the corners of the rectangle contact the circle of radius \(r\) (the radius of the sphere), we have a “constraint equation” which tells us the relationship between the values of \(x\) and \(y\), \(x^2 + y^2 = r^2\).

We could use the equation for the circle to eliminate either \(x\) or \(y\); if we choose to eliminate \(x\), though, we can avoid having to differentiate a square root:

\[
x^2 = r^2 - y^2 \Rightarrow V = 2\pi x^2 y = 2\pi \cdot (r^2 - y^2) \cdot y = 2\pi r^2 y - 2\pi y^3.
\]

(continued)
Since we are looking for the local maximum of this polynomial function for the volume of the cylinder, we can take the derivative of the function for $V$ and set it equal to zero:

$$\frac{d}{dy} V = \frac{d}{dy} (2\pi r^2 y - 2\pi y^3) = 2\pi r^2 - 6\pi y^2 = 0$$

$$\Rightarrow 6\pi y^2 = 2\pi r^2 \quad \Rightarrow y^2 = \frac{1}{3} r^2 \quad \Rightarrow y = r \sqrt[3]{\frac{1}{3}}$$

We ignore the negative square root, since this is a physical measurement.

$$\Rightarrow x^2 = r^2 - y^2 = r^2 - \left(\frac{1}{3} r^2\right) = \frac{2}{3} r^2 \quad \Rightarrow x = r \sqrt[3]{\frac{2}{3}}.$$

So we find that the dimensions of the cylinder with the largest volume that can be inscribed inside the sphere of radius $r$ has a base radius of $x = r \sqrt[3]{\frac{2}{3}} = \left(\frac{\sqrt{6}}{3}\right) \cdot r$ and a height of $h = 2y = 2 \cdot r \sqrt[3]{\frac{1}{3}} = \frac{2\sqrt{3}}{3} \cdot r$. The maximum possible volume for the inscribed cylinder (which is what this Problem asks for) is then

$$V = 2\pi x^2 y = 2\pi \cdot \left(\frac{2}{3} r^2\right) \cdot \left(\sqrt[3]{\frac{3}{\sqrt{3}}} r\right) = \frac{4\sqrt{3}}{9} \pi r^3.$$

Since the volume of the sphere itself is $V_{sph} = \frac{4}{3} \pi r^3$, the inscribed cylinder of greatest volume occupies

$$\frac{V_{cyl}}{V_{sph}} = \frac{\left(\frac{4\sqrt{3}}{9} \pi r^3\right)}{\left(\frac{4}{3} \pi r^3\right)} = \frac{\sqrt{3}/9}{1/3} = \frac{\sqrt{3}}{3} = 0.577$$

of the sphere's volume (this serves as a check that the cylinder has less volume than the sphere and actually fits inside - and a check that we probably solved the problem correctly). The second derivative \( \frac{d^2 V}{dy^2} = -12\pi y < 0 \), since $y$ is positive, so we have found the maximum value for the volume of the cylinder.
The poster, for which we shall define the dimensions length $L$ by width $W$, is to have margins of one inch at the bottom and sides and of two inches at the top, which permits a printed area of $A_p = (L - 3) \cdot (W - 2)$; this is the area we wish to maximize. The poster is to have a total sheet area of $L \cdot W = 180 \text{ in}^2$, so we will be treating the function $A_p = LW - 2L - 3W + 6 = 180 - 2L - 3W + 6 = 186 - 2L - 3W$.

We can use the “constraint equation” for the total area requirement to eliminate one of the variables, say, $W$, to reduce our function to one with a single variable, for which we can find the critical number:

\[
LW = 180 \implies W = \frac{180}{L} \implies A_p = 186 - 2L - 3 \left( \frac{180}{L} \right) = 186 - 2L - \frac{540}{L}
\]

\[
\implies \frac{dA_p}{dL} = -2 - \left( -\frac{540}{L^2} \right) = \frac{540}{L^2} - 2 = 0 \implies \frac{540}{L^2} = 2
\]

\[
\implies L^2 = \frac{540}{2} = 270 \implies L = \sqrt{270} = \sqrt{9 \cdot 30} = 3 \cdot \sqrt{30} \approx 16.4 \text{ in.}
\]

\[
\implies W = \frac{180}{3\sqrt{30}} = \frac{60}{\sqrt{30}} = \frac{60 \cdot \sqrt{30}}{30} = 2 \sqrt{30} \approx 11.0 \text{ in.}
\]

The overall dimensions for the poster should be about 16-½ by 11 inches, leaving a printed area of 13-½ by 9 inches.

Since the length $L$ is a positive number, the second derivative of the printed area function is

\[
\frac{d^2A_p}{dL^2} = \frac{d}{dL} \left( \frac{540}{L^2} \right) = \frac{d}{dL} \left( 540 L^{-2} \right) = -2 \cdot \frac{540}{L^3} < 0,
\]

so the area we have found is a maximum.
A ladder is intended to reach the wall of a building, but it must clear an eight-foot tall fence that stands four feet from that wall. We wish to find the shortest length of ladder required to arrange this; although we don’t prove it, it is reasonable to assume – and in fact correct for optimization – that the ladder must rest on the top of the fence.

We can work out a proportionality using similar triangles to obtain

\[ \frac{x}{8} = \frac{x+4}{H} \implies H = \left(\frac{x+4}{x}\right) \cdot 8 \text{ ft.}, \]

where \( H \) is the height above the ground where the ladder touches the wall (distance not marked on the diagram). The length of the ladder can then be found using the Pythagorean Theorem:

\[ L^2 = H^2 + (x+4)^2 = \left(\frac{x+4}{x}\right)^2 \cdot 8 + (x+4)^2 = (x+4)^2 \cdot \left(1 + \frac{64}{x^2}\right). \]

It is the length \( L \) that we seek to minimize, but since \( L \) is a positive number, it will work just as well to minimize \( L^2 \):

\[ \frac{d}{dx} (L^2) = \frac{d}{dx} \left( (x+4)^2 \cdot \left(1 + \frac{64}{x^2}\right) \right) \]

\[ = 2 \cdot (x+4) \cdot (x+4)' \cdot \left(1 + \frac{64}{x^2}\right) + (x+4)^2 \cdot \left(1 + \frac{64}{x^2}\right)', \]

\[ = 2 \cdot (x+4) \cdot 1 \cdot \left(1 + \frac{64}{x^2}\right) + (x+4)^2 \cdot \left(-2 \cdot \frac{64}{x^3}\right) = 0 \]

\[ \implies 2 \cdot (x+4) \cdot \left(1 + \frac{64}{x^2}\right) = 2 \cdot (x+4)^2 \cdot \left(\frac{64}{x^3}\right) ; \quad \text{(continued)} \]
we can now divide through by \((x + 4)\) and multiply through by \(x^3\), since neither factor can be zero, to simplify this equation to

\[
x^3 \cdot \left(1 + \frac{64}{x^2}\right) = (x + 4) \cdot 64 \quad \Rightarrow \quad x^3 + 64x = 64x + 256
\]

\[
\Rightarrow x^3 = 256 \quad \Rightarrow \quad x = \sqrt[3]{256} \approx 6.35 \text{ ft.}
\]

The foot of the shortest possible ladder would be placed this far from the fence. It is then \(x + 4 \approx 10.35\) feet away from the wall; the top of the ladder is then \(\left(\frac{x + 4}{x}\right) \cdot 8 \approx 13.04\) feet above the ground. The length of the ladder is therefore

\[
L = \sqrt{H^2 + (x + 4)^2} = \sqrt{10.35^2 + 13.04^2} \approx 16.65 \text{ feet}.
\]

44)

This is a variation of the problems about the separation between two moving people or vehicles. One of the boats travels straight south from a dock, so it is convenient to place the origin of a coordinate system at the dock and arrange the axes so that they point north-south and east-west. If we call due north the positive \(y\)-direction, then we can just describe the velocity of this first boat as \(\frac{dy}{dt} = -20 \text{ km/hr.}\); since it starts from the dock at 2:00 PM, which we will call time \(t = 0\), the position of this boat can just be given as \(y = 0 - 20 \cdot t = -20t\). The second boat is headed due east, which we shall call the positive \(x\)-direction, so its velocity is \(\frac{dx}{dt} = +15 \text{ km/hr.}\). This boat’s position can be given simply in terms of \(x\) then; since it arrives at the dock \((x = 0)\) at 3:00 PM \((t = 1 \text{ hour})\), we can find its coordinate function from a point-slope equation: \((x - 0) = 15 \cdot (t - 1) \quad \Rightarrow \quad x = 15t - 15\).
This allows us to use the distance formula to express the separation between the two boats as a function of time: \( s^2 = (15t - 15)^2 + (-20t)^2 \). We are interested in finding when the minimum of the function occurs, so we will look for its critical number. Because the separation \( s \) is a positive number, it will be find to work out the minimum for \( s^2 \):

\[
\frac{d}{dt}(s^2) = \frac{d}{dt}[(15t - 15)^2 + (-20t)^2] = 2 \cdot (15t - 15) \cdot 15 + 2 \cdot (-20t) \cdot (-20) = 0
\]

\[
\Rightarrow 2 \cdot (225t - 225) + 2 \cdot (400t) = 0 \Rightarrow 625t - 225 = 0
\]

\[
\Rightarrow t = \frac{225}{625} = \frac{9}{25} \text{ or } 0.36 \text{ hour }.
\]

Thus, the boats are closest to one another at 0.36 hours = 21.6 minutes after 2:00 PM, or at 2:21:36 PM.

If the question had asked what the minimum separation is at that time, we would find

\[
s^2 = (15 \cdot 0.36 - 15)^2 + (-20 \cdot 0.36)^2 = (-9.6)^2 + (-7.2)^2 = 144 \text{ km}^2
\]

\[
\Rightarrow s = 12 \text{ km}. \text{ The second derivative for } s^2 \text{ is } \frac{d^2}{dt^2}(s^2) = 625 > 0 \text{, so the separation we've found is the minimum.}
\]

The pipeline that is to be run from the oil refinery to the storage tanks on the other side of the river is built so that the overland portion follows the river bank and then crosses beneath the river along a straight path. The river is 2 km. wider and the tanks are 6 km. downriver from the refinery (we won't concern ourselves with the reasons this arrangement occurred...). It costs twice as much ($800,000 per km.) to run the pipeline under the river as it does to run it over land ($400,000 per km.); we wish to find the point at which the pipeline should begin its passage under the river so as to minimize the total cost of construction.
If we place the transition point at a distance \( x \) km. downriver from the refinery, then the portion under the river will follow the hypotenuse of a right triangle with legs equal to the remaining downriver distance, \( 6 - x \) km., and the width of the river, 2 km. The length of the under-river portion of the pipeline is then \( \sqrt{(6-x)^2 + 2^2} \) km. This makes the total cost for constructing the pipeline

\[
C = 4 \cdot x + 8 \cdot \sqrt{(6-x)^2 + 4} \text{ hundred thousand dollars;}
\]

this is the function we want to minimize. Something to keep in mind is that this particular plan for the pipeline limits the domain of the cost function to \( 0 \leq x \leq 6 \) km.; this issue will come up later on.

The critical number is found from

\[
\frac{dC}{dx} = \frac{d}{dx} \left[ 4 \cdot x + 8 \cdot \sqrt{(6-x)^2 + 4} \right] = 4 + \frac{8}{2} \cdot \frac{1}{\sqrt{(6-x)^2 + 4}} \cdot [(6-x)^2 + 4]'
\]

\[
= 4 + 4 \cdot \frac{2 \cdot (6-x) \cdot (-1)}{\sqrt{(6-x)^2 + 4}} = 0 \Rightarrow 4 = 4 \cdot \frac{2 \cdot (6-x)}{\sqrt{(6-x)^2 + 4}}
\]

\[
\Rightarrow \frac{2 \cdot (6-x)}{\sqrt{(6-x)^2 + 4}} = 1 \Rightarrow 2 \cdot (6-x) = \sqrt{(6-x)^2 + 4}
\]

\[
\Rightarrow 2^2 \cdot (6-x)^2 = (6-x)^2 + 4 \Rightarrow 3 \cdot (6-x)^2 = 4 \Rightarrow (6-x)^2 = \frac{4}{3}
\]

\[
\Rightarrow 6 - x = \pm \frac{4}{\sqrt{3}} \Rightarrow x = 6 \pm \frac{4}{\sqrt{3}} \text{ km.}
\]

While the quadratic function appears to leave us with two solutions, the domain of the model cost function for the pipeline excludes a value of \( x \) larger than \( 6 \) km. So the point where the pipeline should begin its run under the river lies \( 6 - \frac{4}{\sqrt{3}} \approx 4.85 \) km. downriver from the refinery.

We were not asked to calculate the minimum cost, which proves to be

\[
C \approx 4 \cdot 4.85 + 8 \cdot \sqrt{(6-4.85)^2 + 4} \approx 37.86 \text{ hundred thousand}
\]

or about 3.79 million dollars. Here is a small demonstration of the importance of optimization: the seemingly simplest plan of just running the pipeline to the point directly opposite the storage tanks and then straight across the river would cost

\[
C_{simple} \approx 4 \cdot 6 + 8 \cdot \sqrt{(6-6)^2 + 4} \approx 40 \text{ hundred thousand}
\]

or 4.00 million dollars.