a) While this limit could be evaluated using L'Hôpital's Rule, it is intended that we solve this problem using the 'cancellation' technique. If we notice that the numerator is a "sum of two cubes", then the ratio can be factored as
\[
\lim_{x \to -3} \frac{x^3 + 27}{x + 3} = \lim_{x \to -3} \frac{x^3 + 3^3}{x + 3} = \lim_{x \to -3} \frac{(x + 3)(x^2 - 3x + 9)}{x + 3} = \lim_{x \to -3} x^2 - 3x + 9 = (-3)^2 - 3(-3) + 9 = 27.
\]
Of course, since we do know L'Hôpital's Rule, we could apply it here, as the limit has the proper indeterminate form:
\[
\lim_{x \to -3} \frac{x^3 + 27}{x + 3} = \frac{0}{0}
\]
\[
= \lim_{x \to -3} \frac{(x^3 + 27)'}{(x + 3)'} = \lim_{x \to -3} \frac{3x^2}{1} = 3(-3)^2 = 27.
\]

b) It is intended that we solve this problem by using the trigonometric Limit Laws we know, specifically that \( \lim_{u \to 0} \frac{\sin u}{u} = 1 \). For our limit, we will first need to factor the ratio suitably:
\[
\lim_{x \to 0} \frac{(\sin 2x)(\sin 3x)(\sin 5x)}{20x^3} = \lim_{x \to 0} \frac{1}{20} \cdot \frac{\sin 2x}{x} \cdot \frac{\sin 3x}{x} \cdot \frac{\sin 5x}{x}.
\]

In order to evaluate these limits, we need to have the argument of the sine function also appear in the denominator; this can be accomplished by multiplying numerator and denominator by the appropriate constant:
\[
\frac{1}{20} \cdot \lim_{x \to 0} \frac{\sin 2x}{x} \cdot (\frac{2}{2}) \cdot \lim_{x \to 0} \frac{\sin 3x}{x} \cdot (\frac{3}{3}) \cdot \lim_{x \to 0} \frac{\sin 5x}{x} \cdot (\frac{5}{5})
\]
\[
= \frac{1}{20} \cdot \lim_{2x \to 0} \frac{\sin 2x}{2x} \cdot 2 \cdot \lim_{3x \to 0} \frac{\sin 3x}{3x} \cdot 3 \lim_{5x \to 0} \frac{\sin 5x}{5x} \cdot 5
\]
\[
= \frac{1}{20} \cdot 1 \cdot 2 \cdot 1 \cdot 3 \cdot 1 \cdot 5 = \frac{3}{2}.
\]

We could also deal with these limits using L'Hôpital's Rule (we would not want to try solving the original limit that way, because of the product in the numerator). For instance,
\[
\lim_{x \to 0} \frac{\sin 2x}{x} = \lim_{x \to 0} \frac{(\sin 2x)'}{2x'} = \lim_{x \to 0} \frac{2 \cos 2x}{1} = \frac{2 \cdot 1}{1} = 2.
\]

Then our limit becomes
\[
\lim_{x \to 0} \frac{(\sin 2x)(\sin 3x)(\sin 5x)}{20x^3} = \frac{1}{20} \cdot \lim_{x \to 0} \frac{\sin 2x}{x} \cdot \lim_{x \to 0} \frac{\sin 3x}{x} \cdot \lim_{x \to 0} \frac{\sin 5x}{x}
\]
\[
= \frac{1}{20} \cdot \lim_{x \to 0} \frac{2 \cos 2x}{1} \cdot \lim_{x \to 0} \frac{3 \cos 3x}{1} \lim_{x \to 0} \frac{5 \cos 5x}{1}
\]
\[
= \frac{1}{20} \cdot 2 \cdot 3 \cdot 5 = \frac{3}{2}.
\]
c) This limit cannot be evaluated by using L'Hôpital's Rule: it is not indeterminate, but undefined as a product, since \( \lim_{x \to 0} \cos \left( \frac{\pi}{x} \right) \) does not exist (there is no value between -1 and 1 which \( \cos \left( \frac{\pi}{x} \right) \) approaches arbitrarily closely as \( x \) approaches zero).

We must instead employ a "Squeeze Theorem" argument:

\[-1 \leq \cos \left( \frac{\pi}{x} \right) \leq +1\]

\[\Rightarrow -\sqrt{x^3} \leq \sqrt{x^3} \cdot \cos \left( \frac{\pi}{x} \right) \leq \sqrt{x^3}\]

For \( x > 0 \)

\[\Rightarrow \lim_{x \to 0^+} -\sqrt{x^3} \leq \lim_{x \to 0^+} \sqrt{x^3} \cdot \cos \left( \frac{\pi}{x} \right) \leq \lim_{x \to 0^+} \sqrt{x^3}\]

\[\Rightarrow 0 \leq \lim_{x \to 0^+} \sqrt{x^3} \cdot \cos \left( \frac{\pi}{x} \right) \leq 0.\]

Thus, by the "Squeeze Theorem", \( \lim_{x \to 0^+} \sqrt{x^3} \cdot \cos \left( \frac{\pi}{x} \right) = 0.\)

2a) In dealing with the question of continuity for a function with a branched definition, we should first check to see whether the function is continuous within each branch.

For \( x < -4 \), \( f(x) = -\frac{1}{(x+3)^2} \); while this function has a discontinuity at \( x = -3 \), this is outside the interval where the function is being used, so this is not a problem. \( f(x) = 2x \) is certainly continuous everywhere, so it is continuous on \(-4 \leq x < 3 \).

For \( x > 3 \), \( f(x) = 6 \cdot (x-2)^{\frac{1}{4}} \) is only defined for \( x > 2 \); since we are using this function in that domain, and it is continuous everywhere in its domain, there is again no problem.

We need only look, then, at the points where the function \( f(x) \) switches from one branch to another. At the first of these, \( x = -4 \), \( f(x) \) is defined by the second branch, \( f(x) = 2x \), so \( f(-4) = 2(-4) = -8 \). It is also clear that

\[\lim_{x \to -4^+} f(x) = \lim_{x \to 4^-} 2x = -8.\]

But, for the left-sided limit, \( \lim_{x \to -4^-} f(x) = \lim_{x \to -4^-} -\frac{1}{(x+3)^2} = -\frac{1}{(-4+3)^2} = -1. \) So \( f(x) \) is not continuous at \( x = -4 \).

In checking the second branch point, \( x = 3 \), we find that \( f(x) \) is defined by the third branch, \( f(x) = 6 \cdot (x-2)^{\frac{1}{4}} \), so \( f(3) = 6 \cdot (3-2)^{\frac{1}{4}} = 6. \) We also see that

\[\lim_{x \to 3^+} f(x) = \lim_{x \to 3^-} f(x) = 6.\]

For the left-sided limit here, \( \lim_{x \to 3^-} 2x = 6 \). So we find that the two-sided limit does exist at \( x = 3 \) and that

\[\lim_{x \to 3} f(x) = f(3) \text{, meaning } f(x) \text{ is continuous at } x = 3.\]

Our function \( f(x) \) is only discontinuous at \( x = -4 \), so its intervals of continuity are \( x < -4 \) and \( x > -4 \), or, in interval notation, \((-\infty, -4), (-4, \infty)\).
b) We have seen that \( \lim_{x \to -4^-} f(x) = -1 \) and \( \lim_{x \to 3^+} f(x) = 6 \). If the second branch of \( f(x) \) becomes \( f(x) = mx + b \) and it is desired to make \( f(x) \) in its entirety continuous, we must arrange for \( \lim_{x \to -4^-} mx + b = -1 \) and \( \lim_{x \to 3^+} mx + b = 6 \). So we must have \\
\( m(-4) + b = -1 \) and \( m(3) + b = 6 \). Subtracting the first equation from the second one gives us \( 3m - (-4)m = 7 - (-1) = 7 \Rightarrow m = 1 \); therefore, \( b = 3 \). So defining the second branch as \( f(x) = x + 3 \) would make \( f(x) \) continuous everywhere.

Check: \( \lim_{x \to -4^-} (x + 3) = -4 + 3 = -1 \neq f(-4) \); \( \lim_{x \to 3^+} (x + 3) = 3 + 3 = 6 = f(3) \).

3) The applicable limit definition of \( f'(x) \) is 
\( f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \).

For our function, \( f(x) = \frac{1}{2} (2x-1)^2 \), this leads to

\[
f'(x) = \lim_{h \to 0} \frac{\frac{1}{2} [2(x+h) - 1]^2 - \frac{1}{2} (2x-1)^2}{h} = \lim_{h \to 0} \frac{\frac{h}{2} [4(x+h)^2 - 4(x+h) + 1] - \frac{h}{2} (4x^2 - 4x + 1)}{h} = \lim_{h \to 0} \frac{4xh + 2h^2 - 2h}{h} = \lim_{h \to 0} 4x + 2h - 2 = 4x - 2.
\]

We can check this result against what we obtain using the Chain Rule:

\[
\frac{d}{dx} \left[ \frac{1}{2} (2x-1)^2 \right] = \frac{1}{2} \cdot 2(2x-1) \cdot \frac{d}{dx} (2x-1) = \frac{1}{2} \cdot 2(2x-1) \cdot 2 = 2(2x-1) = 4x - 2.
\]
For both parts of this problem, we will need to know the slope of a tangent line to this ellipse. We implicitly differentiate the equation for the ellipse with respect to $x$:

$$\frac{d}{dx} \left( 2x^2 - xy + y^2 + 3x - 6y - 5 \right) = \frac{d}{dx} (0)$$

$$\Rightarrow 4x - (y + x \frac{dy}{dx}) + 2y \frac{dx}{dx} + 3 - 6 \frac{dy}{dx} = 0$$

$$\Rightarrow (x - 2y + 6) \frac{dy}{dx} = 4x - y + 3$$

$$\Rightarrow \frac{dy}{dx} = \frac{4x - y + 3}{x - 2y + 6}.$$  

a) At the point $(-2, 2+\sqrt{7})$, then, the slope of the tangent line is

$$\left. \frac{dy}{dx} \right|_{(-2, 2+\sqrt{7})} = \frac{4(-2) - (2+\sqrt{7}) + 3}{(-2) - 2(2+\sqrt{7}) + 6}$$

$$= \frac{-8 - 2 - \sqrt{7} + 3}{-2 - 4 - 2\sqrt{7} + 6} = \frac{-7 - \sqrt{7}}{-2 \sqrt{7}} = \frac{7 + 7\sqrt{7}}{2}.$$  

The equation for the tangent line to that point is

$$(y - [2+\sqrt{7}]) = \left( \frac{7 + 7\sqrt{7}}{2} \right) (x - [-2])$$

$$\Rightarrow y = \left( \frac{7 + 7\sqrt{7}}{2} \right) x + 2 \left( \frac{7 + 7\sqrt{7}}{2} \right) + (2 + \sqrt{7}) = \left( \frac{7 + 7\sqrt{7}}{2} \right) x + (9 + 8\sqrt{7})$$

b) The tangents to this ellipse will be horizontal where $\frac{dy}{dx} = 0$; since the derivative is a quotient, it will be zero where the numerator is zero (provided the denominator is not also zero there). We thus set $4x - y + 3 = 0 \Rightarrow y = 4x + 3$; the tangent points will be at the intersections of this line with the ellipse. If we use this substitution in the equation for the ellipse, we find

$$2x^2 - x(4x+3) + (4x+3)^2 + 3x - 6(4x+3) - 5 = 0$$

$$\Rightarrow 2x^2 - 4x^2 - 3x + 16x^2 + 24x + 9 + 3x - 24x - 18 - 5 = 0$$

$$\Rightarrow 14x^2 - 14 = 0 \Rightarrow x^2 = 1 \Rightarrow x = -1 \Rightarrow y = -1.$$  

The tangent lines are horizontal at $(-1, -1)$ and $(1, 7)$.

The tangents will be vertical where $\frac{dx}{dy}$ is undefined, which will occur where the denominator is zero (provided the numerator is not also zero there). We set $x - 2y + 6 = 0$, yielding $y = \frac{1}{2} x + 3$; these tangent points lie where this line meets the ellipse:

$$2x^2 - x(\frac{1}{2}x+3) + (\frac{1}{2}x+3)^2 + 3x - 6(\frac{1}{2}x+3) - 5 = 0$$

$$\Rightarrow 2x^2 - \frac{1}{2}x^2 - 3x + \frac{1}{4}x^2 + 3x + 9 + 3x - 3x - 18 - 5 = 0$$

$$\Rightarrow \frac{7}{4}x^2 - 14 = 0 \Rightarrow x^2 = 8 \Rightarrow x = -2\sqrt{2} \Rightarrow y = 3 - 2\sqrt{2}.$$  

The tangent lines are vertical at $(-2\sqrt{2}, 3-\sqrt{2})$ and $(2\sqrt{2}, 3+\sqrt{2})$. 
a) We will start by writing \( \frac{1}{4x+1} \) as \((4x+1)^{-1}\). We can look at the first few derivatives to search for a pattern:

\[
f(x) = (4x+1)^{-1} \Rightarrow f'(x) = (-1) \cdot (4x+1)^{-2} \cdot 4 \cdot (4x+1)^{3} = (-1) \cdot 4 \cdot (4x+1)^{-2} \\
f''(x) = (-1) \cdot (4) \cdot (4x+1)^{-3} \cdot 4 = (+1) \cdot (1)(2) \cdot 4^2 \cdot (4x+1)^{-3} \\
f'''(x) = (+1) \cdot (1)(2)(3) \cdot 4^3 \cdot (4x+1)^{-4} \\
\]

What we see is that: 1) the sign alternates with each successive derivative; 2) there is an accumulating product of consecutive integers; 3) there is a power of 4 which increases each time; and 4) the magnitude of the negative exponent of the \((4x+1)\) term is one larger than the level of the derivative. So we can write the general derivative as:

\[
f^{(n)}(x) = (4x+1)^{-n} \cdot \left[ 1 \cdot 2 \cdot 3 \cdots n \right] = (4x+1)^{-n} \cdot \left( \frac{n!}{4^n} \right) \\
\]

The ninth derivative would then be:

\[
f^{(9)}(x) = (-1)^9 \cdot 4^9 \cdot 9! \cdot (4x+1)^{-10} = \frac{4^9 \cdot 9!}{(4x+1)^{10}} \\
\]

b) For an exponential function of the type \( y \cdot e^x \), logarithmic differentiation is really the only useful technique:

\[
y = (\sin x)^x \Rightarrow \ln y = x \cdot \ln(\sin x) \\
\Rightarrow \frac{d}{dx}(\ln y) = \frac{1}{y} \cdot \ln(\sin x) + x \cdot \frac{1}{\sin x} \cdot \frac{d}{dx}(\sin x) \\
\Rightarrow \frac{dy}{dx} = y \cdot \left[ x \cdot \frac{\cos x}{\sin x} \right] = [\ln(\sin x) + x \cot x] \cdot (\sin x)^x \\
\]

6) The expression for the linear approximation of a function \( f(x) \), centered at \( x = a \), is:

\[
f(x) \approx f(a) + f'(a) \cdot (x-a) \\
\]

a) The function is \( f(x) = x^{\frac{1}{4}} \). In using a linear approximation, we prefer to build it upon a value of \( a \) for which \( f(a) \) is easily found. Since \( 81^{\frac{1}{4}} = 3 \) and the choice of \( a = 81 \) is close to the value \( x = 80 \) we wish to make an estimate for, this should give us a fairly precise approximation. We have:

\[
a = 81 \, , \, f(a) = f(81) = 81^{\frac{1}{4}} = 3 \, ; \, f'(x) = \frac{1}{4} x^{-\frac{3}{4}} = \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{16} \cdot (81^{\frac{1}{4}}) \\
\Rightarrow f'(81) = \frac{1}{4} \cdot \frac{1}{16} = \frac{1}{64} \\
\]

The linear approximation for \( f(x) = x^{\frac{1}{4}} \), centered at \( a = 81 \), is then:

\[
f(x) \approx 3 + \frac{1}{64} (x - 81) \\
\]

Our estimate for \( 80^{\frac{1}{4}} \) will be found by setting \( x = 80 \):

\[
f(80) = 80^{\frac{1}{4}} \approx 3 + \frac{1}{64} \cdot (80 - 81) = 3 - \frac{1}{64} = \frac{629}{64} \approx 9.796875 \\
\]

The result from a calculator is \( 80^{\frac{1}{4}} \approx 2.950569 \), so our result is accurate to four decimal places.
b) The function is \( f(x) = \sin x \). The complication in making approximations for trigonometric functions is that the argument must be in "natural" units, which is to say, radians, rather than degrees. We know that \( \sin 30^\circ = \frac{1}{2} \), so we will base our linear approximation on \( a = 30^\circ = \frac{\pi}{6} \); since we want to make an estimate for \( x = 31^\circ \), this will make \( (x-a) = (31^\circ-30^\circ) = 1^\circ = \frac{\pi}{180} \). We have

\[
a = \frac{\pi}{6}, \quad f(a) = f\left(\frac{\pi}{6}\right) = \sin \frac{\pi}{6} = \frac{1}{2};
\]

\[
f'(x) = \frac{d}{dx} (\sin x) = \cos x \Rightarrow f'(\frac{\pi}{6}) = \cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}.
\]

The linear approximation for \( f(x) = \sin x \), centered on \( a = \frac{\pi}{6} \), is

\[
f(x) \approx \frac{1}{2} + \frac{\sqrt{3}}{2} \cdot (x-\frac{\pi}{6}).
\]

So our estimate for \( \sin 31^\circ \) is found by setting \( x = 31^\circ \) or \( (x-a) = 1^\circ = \frac{\pi}{180} \):

\[
f(31^\circ) = \sin 31^\circ \approx \frac{1}{2} + \frac{\sqrt{3}}{2} \cdot \frac{\pi}{180} = \frac{1}{2} + \frac{\pi \sqrt{3}}{360} = \frac{180 + \pi \sqrt{3}}{360} \approx 0.51511...
\]

The result from a calculator is \( \sin 31^\circ \approx 0.515038... \), so our result is accurate to three decimal places.

\( \Box \)

When we are seeking the absolute maxima and minima for a function \( f(x) \) in an interval, we must look at two things: the critical points of \( f(x) \) in the interval, and the values of \( f(x) \) at the endpoints of the interval.

For our function \( f(x) = 2x^3 + 3x^2 - 36x - 21 \) on the interval \([-5, 5]\),

\[
f'(-5) = 2(-5)^3 + 3(-5)^2 - 36(-5) - 21 = 2(-125) + 3(25) + 180 - 21 = -16,
\]

\[
f(5) = 2(5)^3 + 3(5)^2 - 36(5) - 21 = 2(125) + 3(25) - 180 - 21 = 124;
\]

\[
f'(x) = 6x^2 + 6x - 36 = 6(x^2 + x - 6) = 6(x-2)(x+3)
\]

\[\Rightarrow f'(x) = 0 \text{ at } x = -3 \text{ and } x = 2;\]

both of these values are in \([-5, 5]\), so we need to evaluate our function at both values:

\[
f(-3) = 2(-3)^3 + 3(-3)^2 - 36(-3) - 21 = 2(-27) + 3(9) + 108 - 21 = 60,
\]

\[
f(2) = 2(2)^3 + 3(2)^2 - 36(2) - 21 = 2(8) + 3(4) - 72 - 21 = -65.
\]

By comparing these values of \( f(x) \), we see that, on the interval \([-5, 5]\), \((2, -65)\) is both the local and absolute minimum, \((-3, 60)\) is a local maximum, and \((5, 124)\) is the absolute maximum.
a) To find a solution in the interval \([0, 1]\) to the equation \(2e^{-x} = -3 \sin (\pi x + \frac{\pi}{4})\) is equivalent to finding a zero to the function \(f(x) = 2e^{-x} + 3 \sin (\pi x + \frac{\pi}{4})\). This function is a sum of an exponential function and a sine function; each is defined and continuous everywhere, so \(f(x)\) is also. Therefore, our function satisfies the conditions for the Intermediate Value Theorem: since \(f(x)\) is continuous on \((0, 1)\), there exists a value \(x = c\) for which \(f(c)\) has any value between \(f(0)\) and \(f(1)\). We find that
\[
\begin{align*}
f(0) &= 2e^0 + 3 \sin (\pi \cdot 0 + \frac{\pi}{4}) = 2 \cdot 1 + 3 \cdot \frac{\sqrt{2}}{2} > 0 \\
f(1) &= 2e^{-1} + 3 \sin (\pi \cdot 1 + \frac{\pi}{4}) = 2 \cdot \frac{1}{e} + 3 \cdot \left(\frac{-\sqrt{2}}{2}\right) < 0.
\end{align*}
\]
Since \(f(0)\) is positive and \(f(1)\) is negative, the Intermediate Value Theorem tells us that there is a value \(x = c\), where \(0 < c < 1\), for which \(f(c) = 0\) [it does not tell us what that value is]. So there is a value \(x = c\) in the interval \([0, 1]\) for which \(2e^{-x} = -3 \sin (\pi x + \frac{\pi}{4})\).

b) The Mean Value Theorem states that if a function \(f(x)\) is continuous on \([a, b]\) and differentiable on \((a, b)\), then there is a value \(x = c\), where \(a < c < b\), for which the point derivative \(f'(c)\) equals the slope of the secant line from \((a, f(a))\) to \((b, f(b))\). Our function, \(f(x) = \frac{5x}{2x-3}\), is undefined for \(2x-3 = 0 \Rightarrow x = \frac{3}{2}\); this point, however, is not in the interval \([3, 6]\), so we may apply the Mean Value Theorem for this problem. The derivative function is \(f'(x) = \frac{(2x-3) \cdot \frac{5}{2x-3} - (5x) \cdot \frac{1}{2x-3}}{(2x-3)^2} = \frac{-15}{(2x-3)^2}\). The slope of the secant line from \(x = 3\) to \(x = 6\) is given by
\[
m = \frac{f(6) - f(3)}{6 - 3} = \frac{\left(\frac{5 \cdot 6}{2 \cdot 6 - 3}\right) - \left(\frac{5 \cdot 3}{2 \cdot 3 - 3}\right)}{3} = \frac{15 - 5}{3} = \frac{10}{3}.
\]
The value of \(x = c\) for which \(f'(c)\) equals the slope of this line is given by
\[
f'(c) = \frac{-15}{(2c-3)^2} = \frac{-5}{3} \Rightarrow (2c-3)^2 = \frac{15 \cdot 3}{-5} = 27 \Rightarrow 2c-3 = \pm \sqrt{27} = \pm 3\sqrt{3} \Rightarrow c = \frac{3}{2} \pm \frac{3}{2}\sqrt{3}.
\]
Of the two solutions for \(c\), \(c = \frac{3}{2} + \frac{3}{2}\sqrt{3} \approx 4.098\) is in the interval \([3, 6]\); this, then, is the point whose existence is guaranteed by the Mean Value Theorem.
a) The limit \( \lim_{x \to 0} \frac{\cos 6x - \cos 4x}{2x^2} \) leads to the indeterminate ratio \( \frac{0}{0} \). So it is a suitable form for applying L'Hôpital's Rule: 
\[
\lim_{x \to 0} \frac{\cos 6x - \cos 4x}{2x^2} = \lim_{x \to 0} \frac{\frac{d}{dx}(\cos 6x - \cos 4x)}{\frac{d}{dx}(2x^2)} = \lim_{x \to 0} \frac{-6 \sin 6x - (-4 \sin 4x)}{4x}.
\]
This again gives us an indeterminate ratio \( \frac{0}{0} \). We can now proceed in one of two ways:

1. We could apply L'Hôpital's Rule again to obtain
\[
\lim_{x \to 0} \frac{-6 \sin 6x + 4 \sin 4x}{4x} = \lim_{x \to 0} \frac{-36 \cos 6x + 16 \cos 4x}{4} = \frac{-36 \cdot 1 + 16 \cdot 1}{4} = -\frac{20}{4} = -5;
\]

or

2. We could treat this as a trigonometric limit problem, leading to
\[
\lim_{x \to 0} \frac{-6 \sin 6x}{4x} + \frac{4 \sin 4x}{4x} = \lim_{x \to 0} (-6) \cdot \frac{\sin 6x}{6x} \cdot \frac{1}{6} + \lim_{x \to 0} 4 \cdot \frac{\sin 4x}{4x} = (-6) \cdot \frac{3}{2} + 4 \cdot 1 = -9 + 4 = -5.
\]

b) The limit \( \lim_{x \to \infty} \left( \frac{2x-3}{2x+5} \right)^{2x+1} \) leads to the indeterminate power \( 1^\infty \). To bring this into a form suitable for application of L'Hôpital's Rule, we will first need to take the natural logarithm of this function (keep in mind that this will now obtain for us the logarithm of the original limit):
\[
\ln \lim_{x \to \infty} \left( \frac{2x-3}{2x+5} \right)^{2x+1} = \lim_{x \to \infty} \left( \frac{2x-3}{2x+5} \right) \cdot \ln \left( \frac{2x-3}{2x+5} \right),
\]
which now gives the indeterminate product \( \infty \cdot \ln 1 = \infty \cdot 0 \). We will now need to make this into an indeterminate ratio by writing
\[
\lim_{x \to \infty} \frac{\ln \left( \frac{2x-3}{2x+5} \right)}{1} = \lim_{x \to \infty} \frac{\ln \left( \frac{2x-3}{2x+5} \right)}{1},
\]
which at last presents an indeterminate ratio of \( \frac{0}{0} \). We now apply L'Hôpital's Rule to obtain
\[
\lim_{x \to \infty} \left( \frac{\ln \left( \frac{2x-3}{2x+5} \right)}{1} \right)' = \lim_{x \to \infty} \left( \frac{1}{\frac{2x-3}{2x+5}} \cdot \frac{\frac{d}{dx} \left( \frac{2x-3}{2x+5} \right)}{\frac{d}{dx} (2x+1)} \right) = \lim_{x \to \infty} \left( \frac{1}{\frac{2x+5}{2x-3}} \cdot \frac{(2x+5)(2x-3) - (2x-3)(2x+5)}{(2x+5)^2} \right) \cdot (2x+1)^2 - 2
\]
\[
= \lim_{x \to \infty} \left( \frac{2x+5}{2x-3} \cdot \frac{2x-3}{(2x+5)^2} \right) \cdot (2x+1)^2 - 2
\]
\[
= \lim_{x \to \infty} \left( \frac{2x+5}{2x-3} \cdot \frac{2x+1}{2x+5} \right) \cdot \frac{(10 + 6) \cdot (2x+1)^2}{(2x+5)^2} \cdot \frac{(2x+1)^2}{(2x+5)^2} \cdot \frac{(10 + 6)}{(2x+1)^2} = \lim_{x \to \infty} \left( \frac{10 + 6}{2x+1} \right) \cdot \frac{(2x+1)(2x+1)}{(2x-3)(2x+5)}
\]
\[
= -8 \lim_{x \to \infty} \left( \frac{2x+1}{2x-3} \right) \cdot \frac{1}{2x+5} \cdot \frac{1}{2x+5} \cdot \frac{1}{2x+5} = -8 \lim_{x \to \infty} \left( \frac{2 + \frac{1}{x}}{2 - \frac{3}{x}} \right) \cdot \frac{1}{2x+5} \cdot \frac{1}{2x+5} \cdot \frac{1}{2x+5}
\]
\[
= -8 \cdot \frac{2 + \frac{1}{x}}{2 + \frac{3}{x}} = -8.
\]

Recall, however, that this is the natural logarithm of the original limit. So our result is
\[
\lim_{x \to \infty} \left( \frac{2x-3}{2x+5} \right)^{2x+1} = e^{-8}.
\]
Our function is \( g(x) = \frac{x^3 + 1}{x^3 - 1} \).

The domain is all real numbers except where \( g(x) \) is undefined. Since \( g(x) \) is a rational function of polynomials, it is undefined where the denominator is zero (provided the numerator is not also zero there). Since \( x^3 - 1 = 0 \Rightarrow x^3 = 1 \Rightarrow x = 1 \), we find that the domain of \( g(x) \) is all real numbers except \( x = 1 \).

Since the numerator and denominator may be written as \( x^3 + x \), neither polynomial has symmetry. Thus \( g(x) \) has no symmetry either.

The y-intercept for \( g(x) \) is \((0, g(0)) = (0, \frac{0^3 + 1}{0^3 - 1}) = (0, -1)\). The x-intercept(s) are given by \( g(x) = 0 \). Since \( g(x) \) is a rational function, it equals zero when the numerator is zero (provided the denominator is not also zero there). Since \( x^3 + 1 = 0 \Rightarrow x^3 = -1 \Rightarrow x = 1, g(x) \) has a single x-intercept at \((-1, 0)\).

The vertical asymptote occurs where \( g(x) \) is undefined; thus, it is \( x = 1 \). The horizontal asymptote is given by the limit at infinity:

\[
\lim_{x \to \infty} \frac{x^3 + 1}{x^3 - 1} = \lim_{x \to \infty} \frac{1 + \frac{1}{x^3}}{1 - \frac{1}{x^3}} = 1;
\]

so the horizontal asymptote is \( y = 1 \).

The first and second derivatives of \( g(x) \) are

\[
g'(x) = \frac{(x^3 - 1) \frac{d}{dx}(x^3 + 1) - (x^3 + 1) \frac{d}{dx}(x^3 - 1)}{(x^3 - 1)^2} = \frac{(x^3 - 1)(3x^2) - (x^3 + 1)(3x^2)}{(x^3 - 1)^2} = \frac{-6x^2}{(x^3 - 1)^2};
\]

\[
g''(x) = \frac{(x^3 - 1)^3 \frac{d}{dx}(-6x^2) - (-6x^2) \frac{d}{dx}(x^3 - 1)^3}{(x^3 - 1)^4} = \frac{(x^3 - 1)^2 \cdot (-12x) + (6x^2) \cdot 2(x^3 - 1)(3x^2)}{(x^3 - 1)^4}\]
\[
= \frac{-12x^4 + 12x + 36x^4}{(x^3 - 1)^3} = \frac{12x(2x^3 + 1)}{(x^3 - 1)^3}.
\]

We find that \( g'(x) = 0 \) only for \( x = 0 \). Since \( x^2 \) and \( (x^2 - 1)^2 \) can only be positive or zero, \( g'(x) \) can only be negative or zero in the domain of \( g(x) \). So \( g(x) \) can only decrease, except at \( x = 1 \). As for the second derivative, \( g''(x) = 0 \) at \( x = 0 \) or for \( 2x^3 + 1 = 0 \Rightarrow x^3 = -\frac{1}{2} \Rightarrow x = \frac{\sqrt[3]{-2}}{2} \). If we look at the sign of the factors in \( g''(x) \), we see

\[
x < \frac{\sqrt[3]{-2}}{2}, \quad \frac{\sqrt[3]{-2}}{2} < x < 0, \quad 0 < x < 1, \quad x > 1
\]

\[
\begin{array}{cccc}
- & + & + \\
\frac{\sqrt[3]{-2}}{2} < x < 0 & 0 < x < 1 & x > 1 \\
& + & + \\
& + & +
\end{array}
\]

\[
g''(x) < 0 \quad g''(x) > 0 \quad g''(x) < 0 \quad g''(x) > 0
\]

concave downward concave upward concave downward concave upward

There are two special points for this function. At \( x = \frac{\sqrt[3]{-2}}{2} \), \( g'(x) = 0 \) and \( g''(x) = 0 \), so this must be a point of inflection. The same conditions apply at \( x = 0 \), so it is also a point of inflection. Our function \( g(x) \) has no maxima or minima.

How can \( g(x) \) always decrease (except at \( x = 1 \)), yet have a horizontal asymptote? This is due to the peculiar nature of "infinity". Our function \( g(x) \) decrease from close to \( 1 \) for large negative \( x \) to "-oo" at \( x = 1 \), then "starts over" at "+oo" and "continues" to decrease to close to \( 1 \) for large positive \( x \).

A graph of this function is presented in the answer key.
(1) We next establish a relationship between the rate at which the lighthouse beam rotates and the rate at which it sweeps along the shoreline. The perpendicular distance of the lighthouse from the shore is 3 km; the distance along the shoreline from P to the point where the beam is shining at the moment is x.

We can write \( \tan \theta = \frac{x}{3} \). By differentiating implicitly with respect to time \( t \), we obtain

\[
\frac{dx}{dt} \left( \tan \theta \right) = \frac{d}{dt} \left( \frac{x}{3} \right) \Rightarrow \sec^2 \theta \frac{dx}{dt} = \frac{x}{3} \frac{d\theta}{dt}.
\]

So we have \( \frac{dx}{dt} = \frac{3}{\sec^2 \theta} \frac{d\theta}{dt} \). We are told that the lighthouse beam rotates at four revolutions per minute, so \( \frac{d\theta}{dt} = 4 \text{ rev.} \times 2\pi \text{ rad.} \times 60 \text{ sec} = \frac{48\pi}{15} \text{ sec} \). At the moment when \( x = 2 \text{ km} \),

\[
\sec^2 \theta = \left( \frac{3}{\frac{13}{3}} \right)^2 = \frac{2}{3} \times \frac{3}{13} = \frac{2}{13}.
\]

The rate at which the beam is sweeping along the shore at that instant is then \( \frac{dx}{dt} = (3 \text{ km}) \left( \frac{13}{2} \right) \left( \frac{2 \pi \text{ rad.}}{15 \text{ sec}} \right) = \frac{26\pi}{45} \text{ km/sec} = 1.015 \text{ km/sec} \).

(2) We are seeking to minimize the cost of fabricating a cylindrical tank of length \( L \) and radius \( R \) with a hemispherical cap on each end. The material used in making the endcaps cost twice as much per unit area as the material for the cylindrical wall. The two endcaps together would form a spherical shell of radius \( R \) and surface area \( A_{ph} = 4\pi R^2 \). The cylindrical wall has a surface area \( A_{cy} = 2\pi RL \). The cost of materials for the tank is then \( C = 2A_{ph} + A_{cy} = 8\pi R^2 + 2\pi RL \).

The constraint is that the volume of the tank must be \( 400\pi \text{ ft.}^3 \). The volume held by the endcaps is equal to the volume of a sphere of radius \( R \), \( V_{ph} = \frac{4}{3}\pi R^3 \), while the volume of the cylindrical section is \( V_{cy} = \pi R^2 L \). So \( V = V_{ph} + V_{cy} = \frac{4}{3}\pi R^3 + \pi R^2 L = 400\pi \).

To find the minimum of \( C \), we need to reduce it to a function of one variable. We can eliminate \( L \) by solving the constraint equation for it:

\[
\frac{4}{3}\pi R^3 + \pi R^2 L = 400\pi \Rightarrow R^2 L = 400 - \frac{4}{3}\pi R^3 \Rightarrow L = \frac{400 - \frac{4}{3}\pi R^3}{R^2}.
\]

The cost function is then

\[
C = 8\pi R^2 + 2\pi R \left( \frac{400R^2}{3} - \frac{4}{3}\pi R^3 \right) = 8\pi R^2 + \frac{2\pi R^4}{3} - \frac{8\pi R^2}{3} = \frac{16\pi R^2}{3} + \frac{2\pi R^4}{3}.
\]

The critical points may be found by setting \( \frac{dC}{dR} = 0 \) and solving for \( R \):

\[
\frac{dC}{dR} = \frac{d}{dR} \left( \frac{16\pi R^2}{3} + \frac{2\pi R^4}{3} \right) = \frac{32\pi R}{3} - \frac{8\pi R^3}{3} = 0
\]

\[
\Rightarrow \frac{8\pi R}{3} = \frac{8\pi R^3}{3} \Rightarrow R^2 = 8\pi R \times \frac{3}{2\pi} = 75 \Rightarrow R = 75^{\frac{3}{2}} \text{ ft.} = 4.82 \text{ ft.}
\]

The length of the cylindrical section of the tank is then \( L = \frac{400}{(75^{3/2})} \left( \frac{75^{3}}{75} \right) = \frac{400 \cdot 75^{3/2}}{75} = \frac{400 \cdot 75^{3}}{75} = \frac{400 \cdot 75^{3}}{75} = 4.75^{3/2} \text{ ft.} = 16.36 \text{ ft.}
\]

The second derivative of \( C \) is

\[
\frac{d^2C}{dR^2} = \frac{d}{dR} \left( \frac{32\pi R}{3} - \frac{8\pi R^3}{3} \right) = \frac{32\pi}{3} + \frac{24\pi R^2}{3} > 0 \text{ for } R > 0.
\]

So our result is the local minimum for \( C \).
The correspondence between a Riemann sum and a definite integral may be written as

\[
\lim_{n \to \infty} \sum_{i=1}^{n} f(x_i) \Delta x = \lim_{n \to \infty} \left( \frac{b-a}{n} \right) \sum_{i=1}^{n} f(a + i \left( \frac{b-a}{n} \right)) \to \int_{a}^{b} f(x) \, dx,
\]

using rectangles of equal widths and the "right-endpoint" rule. If we compare this Riemann sum to our infinite sum, we see

\[
\lim_{n \to \infty} \left( \frac{b-a}{n} \right) \sum_{i=1}^{n} f(a + i \left( \frac{b-a}{n} \right))
\]

\[
\lim_{n \to \infty} \left( \frac{b-a}{n} \right) \sum_{i=1}^{n} \cos \left( i \cdot \frac{\pi}{4n} \right)
\]

which indicates that \( b - a = \frac{\pi}{4} \), \( f(x) = \cos x \), and \( x_i = a + i \left( \frac{b-a}{n} \right) = i \cdot \frac{\pi}{4n} \).

From the last equation, we find that \( a \) must be zero, since the term containing the index \( i \) is accounted for. The first equation tells us that \( b = a + \frac{\pi}{4} = 0 + \frac{\pi}{4} = \frac{\pi}{4} \). Therefore, our Riemann sum represents the definite integral \( \int_{0}^{\pi/4} \cos x \, dx \), the value of which is

\[
\sin \left( \frac{\pi}{4} \right) - \sin (0) = \frac{\sqrt{2}}{2} - 0 = \frac{\sqrt{2}}{2}.
\]

(14)

\[ a) \quad \int \frac{\sqrt{2y^2 - 5}}{u} \, dy \quad u = 2y^2 - 5 \quad \Rightarrow \quad du = \frac{4y}{\sqrt{2}} \, dy \]

\[
= \frac{1}{4} \int \frac{2y}{\sqrt{2}} \cdot \frac{1}{u} \, dy \quad u + 5 = 2y^2
\]

\[
\rightarrow \int \left[ \frac{1}{4} \left( u + 5 \right)^{2} \cdot \sqrt{u} \cdot \frac{1}{u} \right] \, du
\]

\[
= \frac{1}{4} \int (u + 5)^{2} \cdot \sqrt{u} \, du
\]

\[
= \frac{1}{4} \int u^{\frac{5}{2}} + 2 \cdot 2.5 \cdot u \cdot \sqrt{u} + 25 \cdot \sqrt{u} \, du
\]

\[
= \frac{1}{4} \left( \frac{2}{3} \cdot u^{\frac{7}{2}} + 10 \cdot \frac{2}{5} \cdot u^{\frac{5}{2}} + 25 \cdot \frac{2}{3} \cdot u^{\frac{3}{2}} \right) + C
\]

\[
\rightarrow \frac{1}{56} (2y^2 - 5)^{\frac{7}{2}} + \frac{1}{4} (2y^2 - 5)^{\frac{5}{2}} + \frac{25}{24} (2y^2 - 5)^{\frac{3}{2}} + C
\]

\[ b) \quad \int_{0}^{x} x e^{-3x^2} \, dx \quad u = -3x^2 \quad \Rightarrow \quad du = -6x \, dx \]

\[
\rightarrow \int_{0}^{-\frac{1}{6}} e^{u} \cdot (-\frac{1}{6} \, du)
\]

\[
= -\frac{1}{6} \int_{0}^{-3} e^{u} \, du
\]

\[
= \frac{1}{6} \left( e^{0} - e^{0} \right) = \frac{1}{6} \left( e^{0} - e^{0} \right) = \frac{1}{6} (1 - e^{0})
\]

In solving a definite integral by substitution, transforming the limits along with the integrand will lead us to the same value as we would obtain by then transforming the integral function back to the original variable and using the original limits.
The Fundamental Theorem of (Integral) Calculus tells us that, for a continuous function \( f(x) \) on \([a, b]\),
\[
\frac{d}{dx} \int_a^x f(t) \, dt = f(x), \quad \text{for} \ a < x < b.
\]
If we wish to extend this equation to a situation
where the upper limit of the integration is a function \( u(x) \), we need to use the result from the Theorem
\[
\int_a^x f(t) \, dt = F(x) - F(a), \quad \text{where} \ F(a) = f(a)
\]
\[
\Rightarrow \int_a^{u(x)} f(t) \, dt = F(u(x)) - F(a)
\]
\[
\Rightarrow \frac{d}{dx} \left[ \int_a^{u(x)} f(t) \, dt \right] = \frac{d}{dx} F(u(x)) - \frac{d}{dx} F(a) = \frac{d}{du} F(u) \cdot \frac{du}{dx} - 0 = \left( \frac{dF}{dx} \right)_{u(x)} \cdot \frac{du}{dx} = \frac{dF}{dx} \frac{du}{dx}; \quad \text{this is known as \textit{Lagrange's Rule}}.
\]
For our function, \( F(x) = \int_0^x \tan^2 t \, dt \), we need the constant as the lower limit, so we will reverse the order of integration: \( F(x) = -\int_x^0 \tan^2 t \, dt \). \text{Lagrange's Rule} leads us to
\[
F(x) = \frac{d}{dx} \left[ -\int_x^0 \tan^2 t \, dt \right] = -\tan^2(a^2x) \cdot \frac{d}{dx} (e^{-2x}) = -\tan^2(e^{-2x}) \cdot (-2e^{-2x}) = 2e^{2x} \cdot \tan^2(e^{-2x}).
\]

In order to find the area between the curves for the functions given, we need to know which function is greater (which is the "upper" function and which is the "lower") and the points where the curves intersect in order to set the limits of integration. To find these points, we set the functions equal to one another and solve for \( x \):

\[
f(x) = 2x^2 + 3x^2 - 8x + 12 = g(x) = -3x^3 + 3x^2 + 9x + 30
\]
\[
\Rightarrow 3x^3 + 6x^2 - 15x - 18 = 0 \quad \Rightarrow \quad (x^3 + 2x^2 - 5x - 6) = 0
\]

Without elaborating on the method, we will simply report that the roots of this equation are \( x = -3, \ x = -1, \) and \( x = 2 \); these are the \( x \)-coordinates of the points of intersection of the curves.

To find which curve is the "upper" one in the intervals \(-3 < x < -1\) and \(-1 < x < 2\), we can simply compare values of the functions at a point in each interval:

- \(-3 < x < -1\)
  \[
f(-3) = 2(-3)^3 + 3(-3)^2 - 8(-3) + 12 = -16 + 27 + 24 + 12 = 47
\]
  \[
g(-3) = -(-3)^3 + 3(-3)^2 + 9(-3) + 30 = 8 + 27 - 27 + 30 = 38
\]
  \(\text{So } g(x) > f(x) \text{ on } [-3,-1] \).

- \(-1 < x < 2\)
  \[
f(-1) = 2(-1)^3 + 3(-1)^2 - 8(-1) + 12 = -2 + 3 + 8 + 12 = 25
\]
  \[
g(-1) = -(-1)^3 + 3(-1)^2 + 9(-1) + 30 = 1 + 3 - 9 + 30 = 33
\]
  \(\text{So } f(x) > g(x) \text{ on } [-1,2] \).

The area between the curves is then found by setting up and solving the integral
\[
A = \int_{-3}^{-1} [f(x) - g(x)] \, dx = \int_{-1}^{2} [g(x) - f(x)] \, dx
\]
\[
= \int_{-3}^{-1} (2x^2 + 3x^2 - 15x - 18) \, dx + \int_{-1}^{2} (3x^3 + 6x^2 - 15x - 18) \, dx
\]
\[
= \left[ \frac{2}{3}x^3 + 2x^2 - \frac{15}{2}x^2 - 18x \right]_{-3}^{2} - \left[ \frac{3}{4}x^4 + 2x^3 - \frac{15}{2}x^2 - 18x \right]_{-1}^{2}
\]
\[
= \left[ \frac{3}{4}(-1)^4 + 2(-1)^3 - \frac{15}{4}(-1)^2 - 18(-1) \right] - \left[ \frac{3}{4}(-3)^4 + 2(-3)^3 - \frac{15}{2}(-3)^2 - 18(-3) \right]
\]
\[
= \left[ \frac{3}{4}(1) + 2(-1) - \frac{15}{4}(-1) + 18 \right] - \left[ \frac{3}{4}(81) + 2(-27) - \frac{15}{2}(9) + 54 \right]
\]
\[
= \left[ \frac{3}{4} - \frac{15}{4} + 18 \right] - \left[ \frac{273}{4} - 54 - \frac{135}{2} + 54 \right] = \frac{253}{4}.
\]
The diagram for this solid of revolution is shown at right. The two curves for \( y = x \) and \( y = \sqrt{x} \) intersect at \( x = 0 \) and \( x = 1 \); by setting the two functions equal we find that

\[
x = \sqrt{x} \quad \Rightarrow \quad x^2 = x \quad \Rightarrow \quad x^2 - x = 0 \quad \Rightarrow \quad x(x-1) = 0.
\]

Between \( x = 0 \) and \( x = 1 \), \( \sqrt{x} > x \), so \( y = \sqrt{x} \) will be the "upper" curve and \( y = x \), the "lower" one.

We're rotating the enclosed area about the vertical line \( x = 2 \), so it will be convenient to use the "cylindrical shells" method to integrate the volume of this solid. The "infinitesimal volume" of a shell is \( dV = 2\pi r \cdot (f_{\text{upper}} - f_{\text{lower}}) \cdot dx \); since the axis of rotation is \( x = 2 \) and the enclosed area is between \( x = 0 \) and \( x = 1 \), \( r = 2 - x \). We then have

\[
V = \int_0^1 2\pi (2-x)(\sqrt{x} - x) \, dx = 2\pi \int_0^1 2\sqrt{x} - 2x - x\sqrt{x} + x^2 \, dx
\]

\[
= 2\pi \left[ \frac{2}{3}x^{3/2} - x^2 - \frac{1}{2}x^{3/2} + \frac{1}{3}x^3 \right]_0^1
\]

\[
= 2\pi \left( \frac{2}{3} - 1 - \frac{1}{2} + \frac{1}{3} \right) = \frac{8\pi}{3}.
\]

We can also use the "washer" method if we choose to integrate along the \( y \)-direction. We must re-write the functions in terms of \( y \):

\[
y = \sqrt{x} \quad \Rightarrow \quad x = y^2, \quad y = x \quad \Rightarrow \quad x = y.
\]

The intersection points of these curves are \((0,0)\) and \((1,1)\), so our limits of integration are \( y = 0 \) to \( y = 1 \). The "washer" has an outer radius of \( r_{\text{outer}} = 2 - y \) and an inner radius of \( r_{\text{inner}} = 2 - y \). The "infinitesimal volume" of a "washer" is \( dV = (\pi r_{\text{outer}}^2 - \pi r_{\text{inner}}^2) \cdot dy \).

The volume of our solid is then

\[
V = \int_0^1 \pi \left[ (2-y)^2 - (2-y)^2 \right] \, dy = \int_0^1 4 - 4y^2 + y^4 - 4y + y^2 \, dy
\]

\[
= \int_0^1 y^4 - 5y^2 + 4y \, dy = \pi \left[ \frac{3}{2}y^6 - \frac{5}{3}y^3 + 2y^2 \right]_0^1
\]

\[
= \pi \left( \frac{3}{2} - \frac{5}{3} + 2 + 0 - 0 \right) = \pi \left( -\frac{25}{15} + \frac{30}{15} \right) = \frac{8\pi}{3}.
\]

The average value of a function \( f(x) \) over an interval \([a,b]\) is given by \( \langle f(x) \rangle = \frac{\int_a^b f(x) \, dx}{b-a} \).

For \( f(x) = \frac{2x}{1+x^2} \) on the interval \([0,2]\),

\[
\int_0^2 \frac{2x}{1+x^2} \, dx = \int_0^5 \frac{du}{u} = \ln|u| \bigg|_0^5 = \ln 5 - \ln 1 = \ln 5 = 2 - 0 = 2n5.
\]

\[
\mu = 1 + x^2 \quad \Rightarrow \quad \mu = 2x \cdot \frac{dx}{dx} = \frac{x}{\mu} + \frac{x^2}{\mu} = \frac{x}{\mu} + \frac{x^2}{\mu} = 1.
\]

So the average value of our function over this interval is \( \langle f(x) \rangle = \frac{\ln 5}{2 - 0} = \frac{1}{2} \ln 5 \).

The function takes on this average value at the locations given by

\[
f(x) = \frac{2x}{1+x^2} = \frac{1}{2} \ln 5 \quad \Rightarrow \quad 2x = (\frac{1}{2} \ln 5) + (\frac{1}{2} \ln 5) x^2
\]

\[
\Rightarrow (\frac{1}{2} \ln 5) x^2 - 2x + (\frac{1}{2} \ln 5) = 0,
\]

the solutions to which are \( x = \frac{2 \pm \sqrt{4 - (\ln 5)^2}}{2 \ln 5} \approx 4 \pm 1.3827 \), both of which are in \([0,2]\).