Solutions for Final Exam -- Math 1271, Fall 2001

1. This differentiation is an exercise in using the Chain Rule [note that there is a correction given for this problem on the page facing Page 21]:

\[ \frac{d}{dx} \left( \cos^3(2x^3) \right) = \frac{d}{dx} \left[ \cos(2x^3) \right]^3 = \frac{d}{du} (u^3) \cdot \frac{du}{dx} (2x^3) = 3u^2 \cdot \frac{d}{dx} \left[ \cos(2x^3) \right] \]

\[ = 3u^2 \cdot \frac{d}{dv} (\cos v) \cdot \frac{dv}{dx} (2x^3) \]

\[ = 3u^2 \cdot (-\sin v) \cdot 4x \]

\[ = -12x \cdot \sin(2x^3) \cdot \cos^2(2x^3) \]

2. The Mean Value Theorem states that if a function \( f(x) \) is continuous and differentiable on an interval, then there is a point \( x = c \) where \( f'(c) \) is the same as the average slope of the function over the interval. Our function, \( f(x) = (x+2)^3 \cdot 2^x \), as a product of a polynomial times an exponential function, is certainly continuous and differentiable everywhere. The average slope of \( f(x) \) over the interval \([0,2]\) is

\[ \frac{f(2) - f(0)}{2 - 0} = \frac{(2+2)^3 \cdot 2^2 - (0+2)^3 \cdot 2^0}{2} \]

\[ = \frac{256 - 8}{2} = 124. \] The Mean Value Theorem then guarantees that there is a value for \( c \) in this interval for which \( f'(c) = 124. \)

3. To determine the equation for the tangent line to the curve \( y = x^3 - 2x^2 + 2x + 1 \) at the point \( (2,5) \), we will need to know the value of the function's derivative at \( x = 2 \).

\[ f(x) = x^3 - 2x^2 + 2x + 1 \Rightarrow f'(x) = 3x^2 - 4x + 2 \Rightarrow f'(2) = 3 \cdot (2)^2 - 4(2) + 2 \]

\[ = 12 - 8 + 2 = 6. \] So the slope of the curve at \( x = 2 \) is 6; the equation of the tangent line at \( (2,5) \) in point-slope form is \( (y - 5) = (12 - 8 + 2)(x - 2) = 6(x - 2). \)

4. Since we are asked to find the limit \( \lim_{x \to 3^-} f(x) \) for this branched function, we will need to use the definition for \( x < 3 \). Thus,

\[ \lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} \frac{(x^2 + x - 12)}{(x-3)} = \lim_{x \to 3^-} \frac{(x-3)(x+4)}{(x-3)} = \lim_{x \to 3^-} x + 4 = 7. \]
Since the first derivative, \( f'(x) \), of a function \( f(x) \) indicates its instantaneous rate of change, the function \( f(x) \) is increasing if and only if \( f'(x) > 0 \) and decreasing if \( f'(x) < 0 \).

For the function \( f(x) = x^4 - 6x^2 \), we find that \( f'(x) = 4x^3 - 12x \).

We can then solve the inequalities
\[
f'(x) = 4x^3 - 12x > 0 \Rightarrow 4x(x^2 - 3) > 0
\]
so we must have either
\[
x > 0 \text{ and } x^2 - 3 > 0 \Rightarrow x > 0 \text{ and } x^2 > 3 \Rightarrow x > \sqrt{3}
\]
or \( x < 0 \text{ and } x^2 - 3 < 0 \Rightarrow x < 0 \text{ and } x^2 < 3 \Rightarrow -\sqrt{3} < x < 0
\]

Similarly, \( f'(x) = 4x(x^2 - 3) < 0 \)
\[
\Rightarrow \text{ either } x < -\sqrt{3} \text{ or } 0 < x < +\sqrt{3}.
\]

Thus \( f(x) \) is increasing for \((-\sqrt{3}, 0)\) and \((+\sqrt{3}, \infty)\) and decreasing for \((-\infty, -\sqrt{3})\) and \((0, +\sqrt{3})\). [The choice on the exam includes interval endpoints, which is an arguable decision.]

The second derivative of a function, \( f''(x) \), indicates the rate of change of its slope at a point \( x \). So we have that \( f(x) \) is concave upward if and only if \( f''(x) > 0 \), concave downward if \( f''(x) < 0 \).

For our function, \( f''(x) = 12x^2 - 12 \). If we solve the inequalities, we find
\[
f''(x) = 12x^2 - 12 = 12(x^2 - 1) > 0 \Rightarrow x^2 - 1 > 0
\]
\[
\Rightarrow x^2 > 1 \Rightarrow x > 1 \text{ or } x < -1
\]
and \( f''(x) = 12(x^2 - 1) < 0 \Rightarrow x^2 < 1 \Rightarrow -1 < x < 1
\].

So \( f(x) \) is concave upward over the intervals \((-\infty, -1)\) and \((1, \infty)\) and concave downward for \((-1, 1)\).

If we differentiate the equation \( xy^2 + 3xy + x^2 + 5y = 4 \) implicitly with respect to \( x \), we have
\[
\frac{d}{dx}(xy^2 + 3xy + x^2 + 5y) = \frac{d}{dx}(4)
\]
\[
\Rightarrow (y^2 + x\cdot 2y\cdot \frac{dy}{dx}) + (3y + 3x\cdot \frac{dy}{dx}) + 2x + 5\cdot \frac{dy}{dx} = 0
\]
\[
\Rightarrow (2xy + 3x + 5)\frac{dy}{dx} + (y^2 + 3y + 2x) = 0
\]
\[
\Rightarrow \frac{dy}{dx} = -\frac{y^2 + 3y + 2x}{2xy + 3x + 5}
\]
There are a couple of approaches to differentiating this exponential function. One method is to first change the base of the function to $e$, since we do not have a rule for differentiating exponentials with other bases:

$$4^{3x+2} = (e^{\ln 4})^{3x+2} = e^{(\ln 4) \cdot (3x+2)}$$

We can now differentiate this form of the function with the Chain Rule:

$$\frac{d}{dx} \left[ e^{(\ln 4) \cdot (3x+2)} \right] = \frac{d}{du} (e^u) \cdot \frac{d}{dx} (u)$$

$$= e^u \cdot \frac{d}{dx} \left[ (\ln 4) \cdot (3x+2) \right]$$

$$= e^u \cdot \frac{d}{dx} \left[ (3)(\ln 4) \cdot x + 2(\ln 4) \right]$$

$$= e^u \cdot (3)(\ln 4)$$

$$= (3)(\ln 4) \cdot e^{(\ln 4) \cdot (3x+2)} = (3)(\ln 4) \cdot 4^{3x+2} \cdot (3x+2)$$

We can also use logarithmic differentiation:

$$y = 4^{3x+2} \quad \Rightarrow \quad \ln y = (3x+2) \cdot \ln 4$$

$$\Rightarrow \quad \frac{d}{dx} (\ln y) = \frac{d}{dx} \left[ (\ln 4) \cdot (3x+2) \right]$$

$$= (\ln 4) \cdot \frac{d}{dx} (3x+2)$$

$$\Rightarrow \quad \frac{1}{y} \cdot \frac{dy}{dx} = (\ln 4) \cdot 3$$

$$\Rightarrow \quad \frac{dy}{dx} = 3 \cdot \ln 4 \cdot y = (3 \ln 4) \cdot 4^{3x+2} \cdot (3x+2)$$

If we look at only a small change in the value of $x$, we can use the approximation

$$f'(x) = \frac{dy}{dx} \approx \frac{\Delta y}{\Delta x} \quad \text{The change in the value of the function is then approximately}$$

$$\Delta y \approx f'(x) \Delta x \quad \text{if we start from a particular value } x = a, \text{ then } \Delta x = (x-a),$$

$$\Delta y = f(x) - f(a), \text{ and we can write } f(x) - f(a) \approx f'(a) \cdot (x-a).$$

For our problem, $f(x) = \sqrt[3]{x}$; to estimate $\sqrt[3]{8.3}$, it will be convenient to start from $\sqrt[3]{8} = 2$, so $a = 8$ and $f(a) = 2$. We still need to find $f'(8)$:

$$f'(x) = \frac{d}{dx} (\sqrt[3]{x}) = \frac{d}{dx} (x^{\frac{1}{3}}) = \frac{1}{3} x^{-\frac{2}{3}} = \frac{1}{3\sqrt[3]{x^2}}$$

$$\Rightarrow \quad f'(8) = f'(x)|_{x=8} = \frac{1}{3\sqrt[3]{64}} = \frac{1}{3 \cdot 2^2} = \frac{1}{12}.$$

So our linear approximation is $f(x) \approx f(8) + f'(8) \cdot (x-a) = 2 + \frac{1}{12} (x-8)$, from which we calculate that $f(8.3) = \sqrt[3]{8.3} \approx 2 + \frac{1}{12} (8.3-8) = 2.025$. (from calculator: 2.02459...
For this function \( f(x) \) with its branched definition, each expression involves a polynomial, so the function is continuous on each individual interval. The only potential locations for discontinuities, then, are at the values of \( x \) where the "branches" join, at \( x = 0 \) and \( x = 2 \).

A function \( f(x) \) is continuous at a point \( x = a \) if \( f(a) \) is defined, \( \lim_{x \to a} f(x) \) exists, and \( \lim_{x \to a} f(x) = f(a) \). For \( x = 0 \), the applicable definition is \( f(x) = |x + 2| \) for \( x < 0 \), so \( f(0) = |0 + 2| = 2 \), which is also the limit "from below".

\[ \lim_{x \to 0^-} f(x) = 2 \; \text{the limit "from above" requires the definition of } f(x) \text{ for } 0 < x < 2 \]

and is \( \lim_{x \to 0^+} f(x) = 2 + x^2 = 2 + 0^2 = 2 \); so the two-sided limit \( \lim_{x \to 0} f(x) \) exists and \( \lim_{x \to 0} f(x) = 2 = f(0) \). For \( x = 2 \), the applicable definition is \( f(x) = x^3 \) for \( x \geq 2 \), so \( f(2) = 2^3 = 8 \), which is also the limit "from above".

\[ \lim_{x \to 2^+} f(x) = 8 \; \text{the limit "from below" again requires the definition of } f(x) \text{ for } 0 < x < 2 \]

and is \( \lim_{x \to 2^-} f(x) = 2 + x^2 = 2 + 2^2 = 6 \); hence, a two-sided limit does not exist at \( x = 2 \). Our function \( f(x) \) is thus continuous at \( x = 0 \), but not at \( x = 2 \), so it is continuous everywhere except at \( x = 2 \). □

If we calculate the derivatives of \( f(x) = 4x^3 - x^4 \), we find that \( f'(x) = 12x^2 - 4x^3 = 4x^2(3-x) \) and \( f''(x) = 24x - 12x^2 = 12x(2-x) \).

The critical points of \( f(x) \) are given by \( f'(x) = 4x^2(3-x) = 0 \Rightarrow x = 0 \) and \( x = 3 \). Since \( f''(x) = 12x(2-x) = 0 \), it appears that the point at \( x = 0 \) is neither a local maximum nor a local minimum (it may be a point of inflection). On the other hand, \( f''(3) = 12 \cdot 3 \cdot (2-3) = -36 < 0 \), so the point at \( x = 3 \) is a local maximum. The value of \( f(x) \) at this point is \( f(3) = 4 \cdot 3^3 - 3^4 = 27 \); since \( \lim_{x \to -\infty} 4x^3 - x^4 = -\infty \) and \( \lim_{x \to +\infty} 4x^3 - x^4 = -\infty \), the point at \( x = 3 \) is also an absolute maximum. □
12. For a rational function of polynomials, the vertical asymptotes occur where the denominator is zero, provided the numerator is not also zero there; the horizontal asymptote is the limit at infinity" of the function. For our function, \( f(x) = \frac{(3x-1)^2}{9x^2 - 4} \), the full factorization is \( f(x) = \frac{(3x-1)(3x-1)}{(3x+2)(3x-2)} \). It is plain that the denominator is zero at \( x = -\frac{2}{3} \) and \( x = +\frac{2}{3} \) and that the numerator is not zero for those values, so these are the locations of the vertical asymptotes. To find the horizontal asymptote, we may multiply out the binomial-square in the numerator and evaluate the limit at infinity:

\[
\lim_{x \to \infty} \frac{(3x-1)^2}{9x^2 - 4} = \lim_{x \to \infty} \frac{9x^2 - 6x - 1}{9x^2 - 4} = \frac{3}{9} = \frac{1}{3}.
\]

Using division by \( x^2 \) and L'Hôpital's Rule

Thus, the horizontal asymptote is \( y = 1 \).

13. In order to use logarithmic differentiation, we first need to write the logarithm of

\[
y = f(x) = \frac{3\sqrt{x+1}}{(x+2)\sqrt{x+3}} \Rightarrow \ln y = \ln \left( \frac{3\sqrt{x+1}}{(x+2)\sqrt{x+3}} \right)
\]

\[
= \ln (3\sqrt{x+1}) - \ln (x+2) - \ln (\sqrt{x+3})
\]

\[
= \frac{1}{2} \ln (x+1) - \ln (x+2) - \frac{1}{2} \ln (x+3).
\]

We may now differentiate this equation implicitly with respect to \( x \):

\[
\frac{d}{dx} (\ln y) = \frac{d}{dx} \left[ \frac{1}{2} \ln (x+1) - \ln (x+2) - \frac{1}{2} \ln (x+3) \right]
\]

\[
\Rightarrow \quad \frac{1}{y} \cdot \frac{dy}{dx} = \frac{1}{2} \cdot \frac{1}{x+1} \cdot \frac{d}{dx} (x+1) - \frac{1}{x+2} \cdot \frac{d}{dx} (x+2) - \frac{1}{2} \cdot \frac{1}{x+3} \cdot \frac{d}{dx} (x+3)
\]

\[
\Rightarrow \quad \frac{dy}{dx} = y \cdot \left[ \frac{1}{3} \cdot \frac{1}{x+1} \cdot 1 - \frac{1}{x+2} \cdot 1 - \frac{1}{2} \cdot \frac{1}{x+3} \cdot 1 \right]
\]

\[
= \left( \frac{3\sqrt{x+1}}{(x+2)\sqrt{x+3}} \right) \cdot \left[ \frac{1}{3(x+1)} - \frac{1}{x+2} - \frac{1}{2(x+3)} \right].
\]

We can at last evaluate

\[
f'(1) = \left( \frac{3\sqrt{2}}{(1+2)\sqrt{3}} \right) \cdot \left[ \frac{1}{3(1+1)} - \frac{1}{1+2} - \frac{1}{2(1+3)} \right] = \frac{3\sqrt{2}}{3 \cdot 2} \cdot \left( \frac{1}{6} - \frac{1}{3} - \frac{1}{8} \right) = -\frac{7 \cdot 3\sqrt{2}}{144}
\]
We can solve this initial value problem by integration or even just by using our knowledge of anti-derivatives. Starting from \( f'(x) = x - 4x^3 \), we find the general anti-derivative \( f'(x) = \frac{1}{2}x^2 - x^4 + C \). We now solve an "initial-value problem" to find the arbitrary constant \( C \); we are told that \( f'(1) = 2 \), so

\[
\left. f'(x) \right|_{x=1} = \frac{1}{2} \cdot 1^2 - 1^4 + C = \frac{1}{2} - 1 + C = C - \frac{1}{2} = 2 \Rightarrow C = \frac{5}{2}.
\]

The specific solution for the given condition is then \( f(x) = \frac{1}{2}x^2 - x^4 + \frac{5}{2} \). The general anti-derivative for this function, in turn, is \( f(x) = \frac{1}{6}x^3 - \frac{1}{3}x^2 + \frac{5}{2}x + D \).

We then solve a second initial-value problem to determine \( D \). Since \( f(1) = \frac{1}{6} \), we obtain

\[
\left. f(x) \right|_{x=1} = \frac{1}{6} \cdot 1^3 - \frac{1}{3} \cdot 1^2 + \frac{5}{2} \cdot 1 + D = \frac{1}{6} - \frac{1}{3} + \frac{5}{2} + D = \frac{1}{6}
\]

\[
\Rightarrow D = \frac{1}{5} - \frac{5}{2} = \frac{2 - 25}{10} = -\frac{23}{10}.
\]

Our function is thus \( f(x) = -\frac{1}{5}x^5 + \frac{1}{6}x^3 + \frac{5}{2}x - \frac{23}{10} \).

\(\)

a) The volume of a sphere of radius \( r \) is \( V = \frac{4}{3} \pi r^3 \). We can differentiate this equation implicitly with respect to time to find a relationship between the rate at which the sphere's radius changes and the rate at which its volume changes.

\[
\frac{dV}{dt} = \frac{d}{dt} \left( \frac{4}{3} \pi r^3 \right) \Rightarrow \frac{dV}{dt} = \frac{4}{3} \pi \cdot \frac{dr}{dt} (r^3) \cdot \frac{dr}{dt} = \frac{4}{3} \pi \cdot 3r^2 \cdot \frac{dr}{dt} = 4\pi r^2 \frac{dr}{dt}.
\]

We are told that the volume of the balloon is being increased at a rate \( \frac{dV}{dt} = +100 \text{ cm}^3/\text{sec} \). We can solve the "related-rates" equation now to find the rate at which the radius is changing when the diameter of the balloon is 50 cm. (Or the radius, \( r \), is 25 cm.):

\[
\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt} \Rightarrow \frac{dr}{dt} = \frac{1}{4\pi r^2} \cdot \frac{dV}{dt} = \frac{1}{4\pi \left(25 \text{ cm}^\text{2}\right)} \cdot 100 \text{ cm}^3/\text{sec} = \frac{1}{25\pi} \text{ cm/sec}.
\]

b) We can find a similar related-rates equation for the rate at which the surface area of the sphere is changing. If we implicitly differentiate the surface area equation with respect to time, we obtain

\[
\frac{d}{dt} (A) = \frac{d}{dt} (4\pi r^2) = 4\pi \cdot \frac{dr}{dt} (r^2) \cdot \frac{dr}{dt} = 4\pi \cdot 2r \cdot \frac{dr}{dt} = 8\pi r \frac{dr}{dt}.
\]

So at the moment described in part (a), the surface area is changing at the rate \( \frac{dS}{dt} = 8\pi \left(25 \text{ cm}^\text{2}\right) \cdot \left(\frac{1}{25\pi} \text{ cm/sec}\right) = 8 \text{ cm}^2/\text{sec} \).
a) To find the volume of the solid of revolution using the method of cylindrical shells, we need to add up the infinitesimal volumes given by $dV = 2\pi r \cdot h \cdot dr = 2\pi x \cdot f(x) \, dx$. The integration will be along the $x$-axis, so

$$V = \int_0^5 2\pi x \cdot e^{-x^2} \, dx$$

b) In order to find this volume by the method of disks, we will need to express the function $f(x) = e^{-x^2}$ as a function of $y$:

$$y = e^{-x^2} \Rightarrow \ln y = -x^2$$

$$\Rightarrow x = \pm \sqrt{-\ln y}$$

Since we are rotating the portion of this curve which is in the first quadrant, we will only want the positive square root: $x = \sqrt{-\ln y}$. The infinitesimal volumes of the disks to be added up are $dV = \pi r^2 \, dy = \pi x^2 \, dy$. However, since the integration is along the $y$-axis here, we find that we must set up two integrals, one for the (extremely!) thin layer from $y=0$ to $y = e^{-25}$ for which the radius of the disk extends fully from $x=0$ to $x=5$, and the rest of the solid from $y = e^{-25}$ to $y = 1$, where our function $x = g(y)$ applies. The volume of the solid will then be found from

$$V = \int_0^{e^{-25}} \pi \cdot 5^2 \, dy + \int_{e^{-25}}^1 \pi \cdot (\sqrt{-\ln y})^2 \, dy$$

c) If we evaluate the volume using the integral from the method of shells, we find

$$V = 2\pi \int_0^5 xe^{-x^2} \, dx$$

let $u = -x^2$

then $du = -2x \, dx \Rightarrow x \, dx = -\frac{1}{2} \, du$

$$\Rightarrow 2\pi \int_{-25}^0 e^u \left(-\frac{1}{2} \, du\right) = -\pi \int_{-25}^0 e^u \, du = \pi \int_{-25}^0 e^{u} \, du = \pi \left(e^0 - e^{-25}\right) = \pi (1 - e^{-25})$$

which only differs from $\pi$ starting at the eleventh decimal place.
A rectangle inscribed within a semicircle of radius one will have two of its vertices in contact with the circle. If we call the distance that the rectangle extends from the $y$-axis to the circle $x$ and the height of the rectangle $y$, then the area of the rectangle is $A = 2xy$.

To find $y$ in terms of $x$, we can use the right triangle formed by the height of the rectangle, half of its width, and the radius of the circle, which forms the hypotenuse. This gives us $x^2 + y^2 = 1 \Rightarrow y = \pm \sqrt{1 - x^2}$; we can just use the vertex of the rectangle that lies in the first quadrant, so we will only use the positive square root. So $y = \sqrt{1 - x^2}$ and $A = 2x \cdot \sqrt{1 - x^2}$.

We now look for the critical point of this area function by setting $\frac{dA}{dx} = 0$:

$$\frac{dA}{dx} = \frac{d}{dx} \left[ 2x \sqrt{1-x^2} \right] = 2 \left( \sqrt{1-x^2} + x \cdot \frac{1}{2 \sqrt{1-x^2}} \cdot (-2x) \right) = 0$$

$$\Rightarrow 2 \sqrt{1-x^2} = \frac{x}{2 \sqrt{1-x^2}} \Rightarrow \sqrt{1-x^2} \cdot \sqrt{1-x^2} = \frac{x^2}{2} \cdot \sqrt{1-x^2}$$

$$\Rightarrow 1 - x^2 = \frac{x^2}{2} \Rightarrow 2x^2 = 1 \Rightarrow x = \pm \sqrt{\frac{1}{2}} = \pm \frac{\sqrt{2}}{2} \quad \text{(we will just want the positive square root in the first quadrant)}$$

$$\Rightarrow y = \sqrt{1 - \left(\frac{\sqrt{2}}{2}\right)^2} = \sqrt{1 - \frac{1}{2}} = \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{2}.$$

The rectangle has dimensions $2x \times y = \sqrt{2} \cdot \frac{\sqrt{2}}{2}$, so the area of the largest possible inscribed rectangle is $A = 1$.

**check:** the second derivative is

$$\frac{d^2A}{dx^2} = \frac{d}{dx} \left[ 2 \cdot \frac{1}{2} \cdot \frac{1}{\sqrt{1-x^2}} \cdot (-2x) \right] - \frac{\sqrt{1-x^2} \cdot (4x) - 2x^2 \cdot \frac{1}{4(1-x^2)} \cdot (-2x)}{(\sqrt{1-x^2})^2}$$

$$= -6x (1-x^2) - 2x^3 \quad \text{for} \quad x = \frac{\sqrt{2}}{2}, \quad \frac{d^2A}{dx^2} = \frac{2x (2x^2 - 3)}{(1-x^2)^{3/2}} = \frac{2 \left( \frac{\sqrt{2}}{2} \right) \left( 2 \cdot \frac{1}{2} - 3 \right)}{(1-\frac{1}{2})^{3/2}} = 0,$$

so the critical point is a local maximum.
The Fundamental Theorem of Calculus tells us that if \( f(x) \) is continuous, then
\[
\int_a^b f(x) \, dx = F(b) - F(a)
\]
where \( F(x) \) is the anti-derivative of \( f(x) \). If the upper limit of the integral is a function \( u(x) \), we can write
\[
\int_a^{u(x)} f(x) \, dx = F(u(x)) - F(a); \text{ the derivative of this integral is then}
\]
\[
\frac{d}{dx} \left[ \int_a^{u(x)} f(x) \, dx \right] = \frac{d}{dx} F(u(x)) - \frac{d}{dx} F(a)
\]
\[
= \left[ \frac{d}{du} F(u) \cdot \frac{du}{dx} \right] - 0
\]
\[
= f(u(x)) \cdot \frac{du}{dx}
\]
[Leibniz's Rule]

For our problem with a function \( g(x) \) defined by
\[
g(x) = \int_{\sqrt{x}}^{x^2} \sqrt{t} \sin t \, dt
\]
we need to write it first in the form of integral functions with constant lower limits:
\[
g(x) = \int_{\sqrt{x}}^{a} \sqrt{t} \sin t \, dt + \int_{a}^{x^2} \sqrt{t} \sin t \, dt
\]
\[
= -\int_{a}^{\sqrt{x}} \sqrt{t} \sin t \, dt + \int_{a}^{x^2} \sqrt{t} \sin t \, dt
\]
Applying Leibniz's Rule, then gives us
\[
g'(x) = \frac{d}{dx} \left[ -\int_{a}^{\sqrt{x}} \sqrt{t} \sin t \, dt \right] + \frac{d}{dx} \left[ \int_{a}^{x^2} \sqrt{t} \sin t \, dt \right]
\]
\[
= -\left[ \sqrt{t} \sin t \right]_{t=\sqrt{x}} \cdot \frac{d}{dx} (\sqrt{x}) + \left[ \sqrt{t} \sin t \right]_{t=x^2} \cdot \frac{d}{dx} (x^2)
\]
\[
= -(\sqrt{x} \sin \sqrt{x}) \cdot \frac{1}{2\sqrt{x}} + (\sqrt{x^2} \sin x^2) \cdot 2x
\]
\[
= 2x^2 \sin (x^2) - \frac{1}{2\sqrt{x}} \sin (\sqrt{x})
\]