1. This integral is solved using integration by parts and evaluation as a Type I improper integral.

\[ \int_0^\infty \frac{x^2 e^{-3x}}{x} \, dx \]

\[ u = x^2 \quad v = -\frac{1}{3} e^{-3x} \]

\[ du = 2x \, dx \quad dv = e^{-3x} \, dx \]

\[ = -\frac{1}{3} x^2 e^{-3x} \bigg|_0^\infty - \int_0^\infty (-\frac{1}{3} e^{-3x})(2x \, dx) \]

\[ = \left[ \lim_{t \to \infty} \left( -\frac{1}{3} t^2 e^{-3t} \right) - (-\frac{1}{3} \cdot 0^2 e^{-0}) \right] + \frac{2}{3} \int_0^\infty \frac{x e^{-3x}}{x} \, dx \]

\[ = \frac{2}{3} \int_0^\infty e^{-3x} \, dx = \frac{2}{9} \left[ \lim_{t \to \infty} -\frac{1}{3} e^{-3t} - (-\frac{1}{3} e^{-0}) \right] = \frac{2}{9} \left( 0 + \frac{1}{3} \right) = \frac{2}{27} \]

This integral also requires integration by parts:

\[ w = x \quad z = -\frac{1}{3} e^{-3x} \]

\[ dw = dx \quad dz = e^{-3x} \, dx \]

This limit gives the indeterminate product "\( \infty \cdot 0 \)"

So we must re-write it as

the indeterminate ratio "\( \infty \)"

\[ \lim_{t \to \infty} \left( -\frac{t^2}{3e^{3t}} \right) = \lim_{t \to \infty} \left( \frac{(t^2)'}{(3e^{3t})'} \right) = \lim_{t \to \infty} \frac{-2t}{27e^{3t}} \]

This still gives us an indeterminate ratio "\( \infty \)"

so we apply l'Hopital's Rule again:

\[ \lim_{t \to \infty} \frac{-2t}{27e^{3t}} = \lim_{t \to \infty} \frac{-2}{27e^{3t}} \]

\[ = \frac{2}{27} \]

By an argument similar to the one above, this term is also zero.

\[ \Rightarrow \quad \frac{2}{3} \cdot \frac{1}{3} \int_0^\infty e^{-3x} \, dx = \frac{2}{9} \left[ \lim_{t \to \infty} -\frac{1}{3} e^{-3t} - (-\frac{1}{3} e^{-0}) \right] = \frac{2}{9} \left( 0 + \frac{1}{3} \right) = \frac{2}{27} \]

2. This integral is solved using the method of partial fractions. Since \( (x^2+1) \) is an irreducible

quadrant polynomial, and the numerator of the integrand is a fourth-degree polynomial, while

the denominator would be fifth-degree, we can write

\[ \frac{x^4 + 1}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{x} \]

\[ = Ax^4 + 2Ax^3 + A + Bx^4 + Bx^2 + Cx^3 + Cx + Dx^2 + Ex \]

\[ \Rightarrow \quad \frac{x^4 + 1}{x(x^2+1)^2} \]

\[ \Rightarrow \quad A = 1, \quad B = 0, \quad C = 0, \quad D = -2, \quad E = 0 \]

\[ \Rightarrow \quad \frac{x^4 + 1}{x(x^2+1)^2} = \frac{1}{x} - \frac{2x}{(x^2+1)^2} \]

This integral is solved using \( u = x^2+1 \Rightarrow du = 2x \, dx \)

\[ \int \frac{x^4 + 1}{x(x^2+1)^2} \, dx = \int \frac{1}{x} \, dx - \int \frac{2x}{(x^2+1)^2} \, dx = \ln|x| + \frac{1}{x^2+1} + C \]
3. The function being integrated, $\frac{1}{\sqrt{y-2}}$, is undefined for $y < 2$. So the integral $\int_2^6 \frac{1}{\sqrt{y-2}} \, dy$ is a Type II improper integral. The indefinite integral is solved by substitution:

$$\int \frac{1}{\sqrt{y-2}} \, dy \rightarrow \int \frac{1}{\sqrt{u}} \, du = \int u^{-\frac{1}{2}} \, du = 2u^{\frac{1}{2}} + C \rightarrow 2\sqrt{y-2} + C.$$ 

So $u = y - 2 \Rightarrow du = dy$.

So we find $\int_2^6 \frac{1}{\sqrt{y-2}} \, dy = 2\sqrt{y-2} \bigg|_2^6 = 2\sqrt{6-2} - 2\sqrt{2-2} = 2\sqrt{4} - 2\sqrt{0} = 4$.

4. This is a "powers of trigonometric functions" integral involving the linked functions of tangent and secant. If we separate the integrand as

$$\int \tan^2(2x) \sec^5(2x) \, dx = \int \tan^3(2x) \sec^4(2x) \cdot \tan(2x) \sec(2x) \, dx,$$

then we have the basis for a substitution based on secant. We first replace the "2x" with $u$:

$$u = 2x \Rightarrow du = \frac{1}{2} \, dx \Rightarrow dx = \frac{1}{2} \, du,$$

$$\rightarrow \int \tan^2 u \sec^4 u \cdot \tan u \sec u \cdot (\frac{1}{2} \, du)$$

and then make the substitution for secant, while also applying the variation of the Pythagorean identity, $\tan^2 \theta + 1 = \sec^2 \theta$:

$$v = \sec u \Rightarrow dv = \sec u \tan u \, du; \quad \tan^2 u = \sec^2 u - 1 = v^2 - 1$$

$$\rightarrow \int (v^2 - 1) \cdot v^4 \cdot (\frac{1}{2}) \, dv = \frac{1}{2} \int v^6 - v^4 \, dv = \frac{1}{2} \left( \frac{1}{7} v^7 - \frac{1}{5} v^5 \right) + C$$

$$\rightarrow \frac{1}{14} \sec^7 u - \frac{1}{10} \sec^5 u + C \rightarrow \frac{1}{14} \sec^7(2x) - \frac{1}{10} \sec^5(2x) + C.$$
a) An infinitesimal element of arclength, in rectangular coordinates, is always \( ds = \sqrt{dx^2 + dy^2} \).

Since we are integrating the arclength of this curve along the \( x \)-direction, we will factor out an infinitesimal \( dx \) to obtain \( ds = \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \) \( dx \). For our function \( y = \frac{1}{6} x^3 + \frac{1}{2} x \), the derivative is \( \frac{dy}{dx} = \frac{1}{2} x^2 - \frac{1}{2} x^2 \); along the interval \([1, 2]\), the arclength is given by

\[
\int_1^2 \sqrt{1 + \left( \frac{1}{2} x^2 - \frac{1}{2} x^2 \right)^2} \, dx = \int_1^2 \sqrt{1 + \frac{1}{4} x^4 - \frac{1}{2} + \frac{1}{4} x^4} \, dx
\]

\[
= \int_1^2 \sqrt{\frac{1}{4} x^4 + \frac{1}{2} + \frac{1}{4} x^4} \, dx = \int_1^2 \sqrt{\left( \frac{1}{2} x^2 + \frac{1}{2} x^2 \right)^2} \, dx
\]

\[
= \int_1^2 \frac{1}{2} x^2 + \frac{1}{2} x^2 \, dx = \left( \frac{1}{6} x^3 - \frac{1}{2} x^2 \right) \bigg|_1^2 = \frac{1}{6} (2)^3 - \frac{1}{2} (2)^2 - \frac{1}{6} (1)^3 + \frac{1}{2} \cdot 1
\]

\[
= \frac{8}{6} - \frac{1}{4} - \frac{1}{6} + \frac{1}{2} = \frac{16 - 9 - 12 + 6}{12} = \frac{17}{12}.
\]

b) For a solid of revolution produced by rotation about an axis, the "infinitesimal surface area" is

\[ dA = 2\pi r \, ds = 2\pi r \sqrt{dx^2 + dy^2} \]

If the axis of rotation is the \( x \)-axis, the radius is then \( r = y = f(x) \), the integration will be along the \( x \)-axis, and

\[ dA = 2\pi y \cdot \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \, dx \]

We already know from part (a) that

\[ \sqrt{1 + \left( \frac{dy}{dx} \right)^2} = \frac{1}{2} x^2 + \frac{1}{2} x^2 \],

so our surface area integral over the interval \([1, 2]\) is

\[
\int_1^2 2\pi \cdot \left( \frac{1}{6} x^3 + \frac{1}{2} x \right) \cdot \left( \frac{1}{2} x^2 + \frac{1}{2} x^2 \right) \, dx = 2\pi \int_1^2 \frac{1}{12} x^5 + \frac{1}{12} x + \frac{1}{4} x + \frac{1}{4} x^3 \, dx
\]

\[
= 2\pi \int_1^2 \frac{1}{12} x^5 + \frac{1}{3} x + \frac{1}{4} x^3 \, dx = 2\pi \left( \frac{1}{12} x^6 + \frac{1}{6} x^2 - \frac{1}{8} x^2 \right) \bigg|_1^2
\]

\[
= 2\pi \left[ \frac{1}{12} (2)^6 + \frac{1}{6} (2)^2 - \frac{1}{8} (2)^2 - \frac{1}{12} (1)^6 - \frac{1}{6} (1)^2 + \frac{1}{8} (1)^2 \right]
\]

\[
= \pi \left( \frac{64}{36} + \frac{4}{3} - \frac{1}{16} - \frac{1}{36} - \frac{1}{3} + \frac{1}{4} \right) = \pi \left( \frac{256 + 96 - 9 - 4 - 48 + 36}{144} \right) = \frac{423\pi}{144} = \frac{47\pi}{16}.
\]
a) The first thing that will be useful is to calculate the centroid of a semicircle of radius $R$ and of a triangle of base $b$ and height $H$.

At a height $y$ above its base, the width of a semicircle is given by $w(y) = 2\sqrt{R^2 - y^2}$. For a semicircular plate of uniform density, the $y$-coordinate of the centroid is given by

$$\bar{y} = \frac{\int_0^R y \cdot w(y) \, dy}{\int_0^R w(y) \, dy} = \frac{\int_0^R y \cdot 2\sqrt{R^2 - y^2} \, dy}{\int_0^R 2\sqrt{R^2 - y^2} \, dy} = \frac{\int_0^R y \cdot 2\sqrt{R^2 - y^2} \, dy}{\int_0^R 2\sqrt{R^2 - y^2} \, dy}$$

$$= \frac{2}{3} R^3$$

$$= \frac{3}{2} R^3 - \frac{4}{3} R^3$$

For the triangle, the width as a function of height $y$ above the base is $w(y) = b - \left(\frac{b}{H}\right) y$. Thus,

$$\bar{y} = \frac{\int_0^b y \cdot \left[b - \left(\frac{b}{H}\right) y\right] \, dy}{\int_0^b \left[b - \left(\frac{b}{H}\right) y\right] \, dy} = \frac{\int_0^b by - \left(\frac{b}{H}\right)y^2 \, dy}{\int_0^b b - \left(\frac{b}{H}\right)y \, dy} = \frac{\frac{1}{2}by^2 - \frac{1}{2}\left(\frac{b}{H}\right)y^3}{\frac{1}{2}by - \frac{1}{2}\left(\frac{b}{H}\right)y^3} = \frac{\frac{1}{2}b^3}{\frac{1}{2}b^2} = \frac{b}{2}.$$

The ice-cream cone sign is a composite object made up of a semi-circular plate with radius $R = 0.5$ m. and density $\rho_s = 1.5 \rho$ and a triangular plate with height $H = 1.5$ m., base $b = 1.0$ m., and density $\rho_t = \rho$. The mass of the semi-circle is then $M_s = \rho_s \cdot \frac{1}{2} \pi R^2 = 1.5 \rho \cdot \frac{1}{2} \pi \cdot (0.5)^2 = \rho \cdot \frac{3}{16} \pi$ and the mass of the triangle is $M_t = \rho_t \cdot \frac{1}{2}bH = \rho \cdot \frac{1}{2} \cdot (1.0) \cdot (1.5) = \rho \cdot \frac{3}{4}$. If we set $y = 0$ as the line where the plates of the plates are joined, then the centroid of the semi-circle lies at $\bar{x}_s = \frac{4}{3\pi} R = \frac{4}{3\pi} \cdot (0.5) = \frac{2}{3\pi} \text{ m.}$ and the centroid of the triangle is located at $\bar{x}_t = -\frac{1}{3} H = -\frac{1}{3} \cdot (1.5) = -\frac{1}{2} \text{ m.}$ The centroid for the entire composite structure is given by

$$\bar{y} = \frac{M_s \bar{y}_s + M_t \bar{y}_t}{M_s + M_t} = \frac{\left(\rho \cdot \frac{3}{16} \pi \cdot \frac{2}{3\pi}\right) + \left(\rho \cdot \frac{3}{4}\right) \left(-\frac{1}{2}\right)}{\rho \cdot \frac{3}{16} \pi + \rho \cdot \frac{3}{4}} = \frac{\frac{4}{3} - \frac{3}{2}}{\frac{3}{16} + \frac{3}{4}} = -\frac{4}{3\pi + 12} \text{ m.},$$

or about 0.7 cm. below the line where the plates join and on the vertical symmetry axis.

b) Pappus' Theorem tells us that the volume of a solid of revolution is equal to the area of the figure revolved about the axis times the circumference of the circle swept out by the centroid. Since the base of the semi-circle is vertical here, the centroid will be $\frac{4}{3\pi} R$ further from the axis than the base is, or $2 \cdot \frac{4}{3\pi} (0.5) \text{ m.}$ away from the axis. The area of the semi-circle is $\frac{1}{2} \pi R^2 = \frac{1}{2} \pi (0.5)^2 \text{ m}^2$. So the volume of the solid is $V = A \cdot 2\pi r = \left(\frac{1}{2} \pi \cdot \frac{1}{2}\right) \cdot (2\pi) \cdot (2 + \frac{4}{3\pi} \cdot 0.5) \text{ m}^3 = \frac{\pi (3\pi \cdot 12)}{6} \text{ m}^3 = 5.46 \text{ m}^3$. 

\[ \text{Diagram of ice-cream cone sign} \]
In analyzing a mixing problem, it is the most convenient to set up a rate equation describing the rate at which the quantity of a material inside a container changes in terms of the rates at which it is brought in and taken out. Because the rates at which salt is being transferred is expressed in terms of rates of fluid flow and concentrations of salt, we construct our rate equation as follows:

$$\frac{dm}{dt} = \frac{d\text{mass brought in}}{dt} - \frac{d\text{mass taken out}}{dt}$$

$$\Rightarrow \frac{d}{dt} \left[ V_{\text{tank}} \cdot c(t) \right] = \frac{d}{dt} \left[ V_{\text{in}}(t) \cdot c_{\text{in}} \right] - \frac{d}{dt} \left[ V_{\text{out}}(t) \cdot c_{\text{out}} \right]$$

$$\Rightarrow V_{\text{tank}} \frac{dc}{dt} = c_{\text{in}} \cdot \frac{dV_{\text{in}}}{dt} - c_{\text{out}} \cdot \frac{dV_{\text{out}}}{dt}.$$ 

At this point, we may apply the information from the problem:

$$(1000 \text{ L}) \cdot \frac{dc}{dt} = \left[ (5 \text{ L/min}) \cdot (0.02 \text{ kg/L}) \right] + \left[ (10 \text{ L/min}) \cdot (0.05 \text{ kg/L}) \right] \text{ inflow from left-hand pipe} - \left[ (15 \text{ L/min}) \cdot c(t) \right] \text{ inflow from right-hand pipe}.$$ 

$$\Rightarrow 1000 \cdot \frac{dc}{dt} = 0.5 - 15 \cdot c(t)$$

We now have a separable differential equation, which we will re-write as:

integrate both sides\[\int \frac{dc}{0.5 - 15c}\]The integration can be performed using a u-substitution: $u = 0.5 - 15c \Rightarrow du = -15 \, dc \Rightarrow dc = -\frac{1}{15} \, du$

$$\int \frac{-\frac{1}{15} \, du}{u} = \int \frac{1}{1000} \, dt \quad \Rightarrow \quad -\frac{1}{15} \ln|u| = \frac{1}{1000} \, t + C \quad \Rightarrow \quad \ln|u| = -\frac{15}{1000} \, t + C'$$

This would be a good point at which to apply the initial condition of the situation, that at $t=0$, the salt concentration in the tank is $c(0) = 0.02 \text{ kg/L}$. That gives us $u(0) = 0.5 - 15(0.02) = 0.5 - 0.3 = 0.2$; hence, $\ln 0.2 = C'$. Our equation is now

$$\ln (0.5 - 15c(t)) = -\frac{15}{1000} \, t + \ln 0.2 \quad \Rightarrow \quad e^{\ln (0.5 - 15c(t))} = e^{\frac{15}{1000} \, t + \ln 0.2}$$

$$\Rightarrow 0.5 - 15c(t) = e^{-\frac{15}{1000} \, t} \cdot e^{\ln 0.2} \quad \Rightarrow \quad 0.5 - 15c(t) = 0.2 \cdot e^{-\frac{15}{1000} \, t}$$

$$\Rightarrow 15c(t) = 0.5 - 0.2 \cdot e^{-\frac{15}{1000} \, t} \quad \Rightarrow \quad c(t) = \frac{1}{30} - \left( \frac{1}{15} \right) \cdot e^{-\frac{15}{1000} \, t} \text{ kg/L}.$$ 

The time unit used is minutes, so the concentration of salt in the tank after one hour (60 minutes) is $c(60) = \frac{1}{30} - \left( \frac{1}{15} \right) \cdot e^{-\frac{15}{1000} \cdot 60} = \frac{1}{30} - \left( \frac{1}{75} \right) \cdot e^{-0.9} \approx 0.029 \text{ kg/L}$. The long-term result for the concentration is given by the "limit at infinity"; hence, we find

$$\lim_{t \to \infty} c(t) = \lim_{t \to \infty} \frac{1}{30} - \left( \frac{1}{15} \right) \cdot e^{-\frac{15}{1000} \, t} = \frac{1}{30} - \left( \frac{1}{75} \right) = 0.033 \text{ kg/L}.$$ 

The mass of salt in the tank at any time can be found by multiplying $c(t)$ by $V_{\text{tank}} = 1000 \text{ L}$:

$$m(t) = V_{\text{tank}} \cdot c(t) = 1000 \left[ \frac{1}{30} - \left( \frac{1}{75} \right) \cdot e^{-\frac{15}{1000} \, t} \right] = \frac{100}{3} - \left( \frac{40}{3} \right) \cdot e^{-\frac{15}{1000} \, t} \text{ kg}.$$
The removal of many foreign substances from a biological organism occurs in a manner similar to exponential decay, that is, the rate at which such a substance leaves the body is proportional to the amount presently within it. The rate of change in the mass of the dye tracer in the patient's body is described by \( \frac{dm}{dt} = -km \), \( k > 0 \). The solution to this separable differential equation is

\[
\int \frac{dm}{m} = \int -k \, dt \Rightarrow \ln |m| = -kt + C \Rightarrow m = e^{-kt+C} = e^{-kt} \cdot e^C = Ae^{-kt}.
\]

We can now apply the information from the problem. We are told that the initial mass of tracer is 100 mg., so \( m(0) = Ae^{-k \cdot 0} = A = 100 \). In three days, we find \( m(3) = 100 \cdot e^{-k \cdot 3} = 30 \Rightarrow e^{-3k} = \frac{30}{100} \Rightarrow -3k = \ln \frac{30}{100} \Rightarrow k = -\frac{1}{3} \ln \frac{30}{100} = \frac{1}{3} \ln \frac{3}{30} \approx 0.4013 \).

a) The "half-life" \( T \) for the dye tracer is the length of time required for the mass within the patient to be reduced by one-half. Thus,

\[
m(T) = 100 \cdot e^{-kT} = \frac{1}{2} \cdot m(0) = \frac{1}{2} \cdot 100 = 50 \Rightarrow e^{-kT} = \frac{50}{100} = \frac{1}{2} \]

\[
\Rightarrow -kT = \ln \frac{1}{2} = -\ln 2 \Rightarrow T = \frac{\ln 2}{k} = \frac{\ln \left( \frac{100}{50} \right)}{\frac{1}{3} \ln \frac{3}{30}} \approx 1.73 \text{ days}.
\]

This is a general relationship between the half-life and the decay constant.

b) In one week, the mass of dye tracer remaining in the patient's body is

\[
m(7) = 100 \cdot e^{-7k} = 100 \cdot e^{-7 \left( \frac{1}{3} \ln \frac{30}{100} \right)} = 100 \cdot \left( e^{\ln \frac{30}{100}} \right)^{-\frac{7}{3}} = 100 \cdot \left( \frac{30}{100} \right)^{-\frac{7}{3}} \approx 6.03 \text{ mg}.
\]

The slope of a tangent line at the point \((x,y)\) on a parametric curve \( x = f(t), \ y = g(t) \) is given by

\[
\frac{dy}{dx} = \frac{dy/dt}{dx/dt} \quad \text{or} \quad \frac{dy}{dt} \cdot \frac{1}{dx/dt}.
\]

For our curve, \( x = 3t - t^3, \ y = 3t^2 \Rightarrow \frac{dy}{dx} = \frac{6t}{3-3t^2} = \frac{6t}{3(1-t^2)} \).

a) A tangent line to this curve is horizontal where the numerator in \( \frac{dy}{dx} \) is zero (provided the denominator is not also zero there). We have \( 6t = 0 \Rightarrow t = 0 \), so the tangent is horizontal at \((x,y) = (3 \cdot 0 - 0^3, 3 \cdot 0^2) = (0,0)\).

A tangent line is vertical where the denominator in \( \frac{dy}{dx} \) is zero (provided the numerator is not also zero there). We have \( 3-3t^2 = 3(1-t^2) = 0 \Rightarrow t^2 = 1 \Rightarrow t = \pm 1 \).

There are vertical tangents at two points, \( t = -1: (x,y) = (3 \cdot (-1) - (-1)^3, 3 \cdot (-1)^2) = (-2,3) \)

\( t = +1: (x,y) = (3 \cdot 1 - 1^3, 3 \cdot 1^2) = (2,3) \).
b) We need to find the value of the parameter $t$ at the point $(-18, 27)$ in order to use our expression for $\frac{dy}{dx}$. We have $y = 3t^2 - 27 \Rightarrow t^2 = 9 \Rightarrow t = \pm 3$; but also, $x = 3t - t^3 = -18$, for which only $t = 3$ works ($t = -3 \Rightarrow y = 3[-3] - [-3]^3 = 18$).

The slope of the tangent line there is $\frac{dy}{dx}_{t=3} = \frac{6 \cdot 3 - 3 \cdot 9}{3 \cdot 3^2} = \frac{18}{-27} = -\frac{2}{3}$. In point-slope form, the equation for the tangent line at $(-18, 27)$ is $(y - 27) = -\frac{2}{3}(x + 18) \Rightarrow y = -\frac{2}{3}x + \frac{27}{2}$ in slope-intercept form.

c) The general appearance of this parametric curve is shown at right, but it is not necessary to graph it. We are told that there is a loop, so there must be a point where the curve crosses itself, that is to say, there is some point which corresponds to two different values of $t$.

We note that $y = 3t^2$ is an even function, meaning that a value of $y$ produced by $t$ is also produced by $(-t)$. So we will now look for the value of $x$ produced by both $t$ and $(-t)$:

\[ x = 3t - t^3 = 3(-t) - (-t)^3 \Rightarrow 3t - t^3 = -3t + t^3 \Rightarrow 6t - 2t^3 = 0 \]
\[ \Rightarrow 2t(3-t^2) = 0. \] So either $t=0$, which we already know about, or $3-t^2=0 \Rightarrow t^2=3 \Rightarrow t = \pm \sqrt{3}$, which gives us the crossing point $(3 \cdot \sqrt{3} - [\sqrt{3}]^3, 3 \cdot [\sqrt{3}]^2)$ = $(0, 9)$.

As for the length of this loop, we will now want to integrate the arc length from $t = -\sqrt{3}$ to $t = +\sqrt{3}$. Hence,

\[ S = \int ds = \int \sqrt{dx^2 + dy^2} = \int_{\sqrt{3}}^{\sqrt{3}} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt = \int_{\sqrt{3}}^{\sqrt{3}} \sqrt{(3-3t^2)^2 + (6t)^2} dt \]

\[ = \int_{\sqrt{3}}^{\sqrt{3}} \sqrt{9 - 18t^2 + 9t^4 + 36t^2} dt = \int_{\sqrt{3}}^{\sqrt{3}} \sqrt{9 + 18t^2 + 9t^4} dt \]
\[ = \int_{\sqrt{3}}^{\sqrt{3}} \sqrt{(3+3t^2)^2} dt = \int_{\sqrt{3}}^{\sqrt{3}} 3 + 3t^2 dt \]
\[ = 3t + t^3 \bigg|_{-\sqrt{3}}^{\sqrt{3}} = 3(\sqrt{3})^2 + [\sqrt{3}]^3 - 3(-\sqrt{3}) - [\sqrt{3}]^3 \]
\[ = 3\sqrt{3} + 3\sqrt{3} + 3\sqrt{3} + 3\sqrt{3} = 12\sqrt{3}. \]
The general expression for the area between two polar curves is
\[
\int \frac{1}{2} r_2^2 - \frac{1}{2} r_1^2 \, d\theta.
\]
Just as with the integration for the area between two curves in rectangular coordinates, we must use this formula with caution.

First of all, the circle and the limaçon cross one another, so the roles of "outer" and "inner" curve change over the course of a cycle. For the right-hand crescent in the graph, the circle is on the outside, while the limaçon is outside for the left-hand crescent.

Since the curves are symmetrical about the \( x \)-axis, we can integrate just the half of the crescents above the axis and double the resulting areas. The intersection point of the two curves is given by
\[
4 \cos \theta = 2 + \cos \theta \Rightarrow 3 \cos \theta = 2 \Rightarrow \cos \theta = \frac{2}{3}. \quad \text{We will call this angle} \\
\Phi = \cos^{-1} \left( \frac{2}{3} \right), \quad \text{which defines one limit for our integrations.}
\]

For the right-hand crescent, we may start the integration for both curves at \( \theta = 0 \).

The area of this crescent is then given by
\[
A_{rc} = 2 \int_0^{\Phi} \frac{1}{2} (4 \cos \theta)^2 - \frac{1}{2} (2 + \cos \theta)^2 \, d\theta = \int_0^{\Phi} 16 \cos^2 \theta - 4 - 4 \cos \theta - \cos^2 \theta \, d\theta
\]
\[
= \int_0^{\Phi} 15 \cos^2 \theta - 4 \cos \theta - 4 \, d\theta = \int_0^{\Phi} 15 \cdot \frac{1}{2} (1 + \cos 2\theta) - 4 \cos \theta - 4 \, d\theta
\]
\[
= \int_0^{\Phi} \frac{15}{2} \cos 2\theta - 4 \cos \theta + \frac{7}{2} \, d\theta = \left( \frac{15}{4} \sin 2\theta - 4 \sin \theta + \frac{7}{2} \theta \right)_0^{\Phi}
\]
\[
= \frac{15}{4} \sin 2\Phi - 4 \sin \Phi + \frac{7}{2} \Phi - 0 + 0 - 0 = \frac{15}{2} \sin \Phi \cos \Phi - 4 \sin \Phi + \frac{7}{2} \Phi.
\]

The angle \( \Phi \) is in the first quadrant and \( \cos \Phi = \frac{2}{3} \), so \( \sin \Phi = \sqrt{1 - \cos^2 \Phi} = \sqrt{1 - \left( \frac{2}{3} \right)^2} = \sqrt{\frac{5}{9}} = \frac{\sqrt{5}}{3} \).

Our result for the right-hand crescent is thus
\[
A_{rc} = \frac{15}{2} \cdot \Phi \cdot \frac{\sqrt{5}}{3} - 4 \cdot \frac{\sqrt{5}}{3} + \frac{7}{2} \cos^{-1} \left( \frac{2}{3} \right) = \left( \frac{5}{3} - \frac{4}{3} \right) \sqrt{5} + \frac{7}{2} \cos^{-1} \left( \frac{2}{3} \right) = \frac{\sqrt{5}}{3} + \frac{7}{2} \cos^{-1} \left( \frac{2}{3} \right).
\]
\[
= 3.689
\]

(continued)
In performing the integration for the left-hand crescent, we must be more careful.
The limaçon only has positive radii, so it crosses the x-axis at $\Theta = \pi$. However, the circle
can have “negative radii”, so it crosses the x-axis at the origin, where $\Theta = \frac{\pi}{2}$
for this curve. Thus we need to integrate the two curves over different intervals.
Because the limaçon is now the “outer” curve, we have

$$A_{gc} = 2 \int_{\pi}^{\pi} \frac{1}{2} (2+\cos \Theta)^2 \, d\Theta - 2 \int_{\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2} (4\cos \Theta)^2 \, d\Theta$$

$$= \int_{\frac{\pi}{2}}^{\pi} 4 + 4\cos \Theta + \cos^2 \Theta \, d\Theta - \int_{\frac{\pi}{2}}^{\frac{\pi}{2}} 16\cos^2 \Theta \, d\Theta$$

$$= \int_{\frac{\pi}{2}}^{\pi} 4 + 4\cos \Theta + \frac{1}{2} (1 + \cos 2\Theta) \, d\Theta - \int_{\frac{\pi}{2}}^{\frac{\pi}{2}} 16 \cdot \frac{1}{2} (1 + \cos 2\Theta) \, d\Theta$$

$$= \int_{\frac{\pi}{2}}^{\pi} \frac{9}{2} + 4\cos \Theta + \frac{1}{2} \cos 2\Theta \, d\Theta - \int_{\frac{\pi}{2}}^{\frac{\pi}{2}} 8 + 8\cos 2\Theta \, d\Theta$$

$$= \left[ \frac{3}{2} \Theta + 0 - \frac{9}{2} \sin \Theta - 4\sin \Theta \right]_{\frac{\pi}{2}}^{\pi} - \left[ 8\Theta + 4\sin 2\Theta \right]_{\frac{\pi}{2}}^{\pi}$$

$$= \left( \frac{3}{2} \pi + 0 - \frac{9}{2} \sin \pi - 4\sin \pi \right) - \left( 8\pi + 4\sin 2\pi \right)$$

$$= \frac{3}{2} \pi + 0 - \frac{9}{2} \sin \frac{\pi}{2} - 4\sin \frac{\pi}{2} - \left( 8\pi + 4\sin 2\pi \right)$$

$$= \frac{3}{2} \pi + 0 - \frac{9}{2} \cdot 1 - 4 \cdot 0 - 8\pi - 4\sin 2\pi$$

$$= \frac{3}{2} \pi + 0 - \frac{9}{2} + 0 - 8\pi - 4\sin 2\pi$$

This is nearly identical to the result for the right-hand crescent, except for the additional
term of $\frac{\pi}{2}$. So we can write $A_{gc} = \frac{3}{2} \pi + \frac{15}{2} + 1\cos^{-1}(\frac{3}{2}) = 5.260.$

1. We need to take this equatin for a conic section in general form and re-write it in standard form.
   We can manage this by using a “completing the squares” technique. We will first group the
terms involving x or involving y and then extract any common factors:

$$16x^2 + 64x - 9y^2 - 30y = 305 \Rightarrow 16 \left( x^2 + 4x \right) - 9 \left( y^2 + 10y \right) = 305$$

We then create binomial squares by adding appropriate constant terms to the equation:

$$16 \left( x^2 + 4x + \frac{4}{4} \right) - 9 \left( y^2 + 10y + \frac{25}{4} \right) = 305 + 16 \cdot 4 + (-9) \cdot 25$$

$$\Rightarrow 16 \left( x + 2 \right)^2 - 9 \left( y + 5 \right)^2 = 305 + 64 - 225 = 144$$

Finally, we divide through by 144 to put the equation in standard form.

$$\frac{16 \left( x + 2 \right)^2}{144} - \frac{9 \left( y + 5 \right)^2}{144} = 1 \Rightarrow \frac{(x+2)^2}{9} - \frac{(y+5)^2}{16} = 1$$

This conic section is a hyperbola with its center at $(-2, -5)$. The positive term is the
one involving x, so the main symmetry axis of the hyperbola is parallel to the x-axis. The
lager denominator is associated with the “y-term”, so the semi-major axis $a = 4$, is parallel to
the x-axis; the semi-minor axis $b = 3$ is parallel to the x-axis. The focal distance of
a hyperbola is given by $c^2 = a^2 + b^2 = 4^2 + 3^2 = 25 \Rightarrow c = 5$.

The vertices of our hyperbola are located at $(-2 \pm 3, -5) = (-2 \pm 3, -5)$.
The two lie at $(-2 \pm 5, -5) = (-2 \pm 5, -5)$. The asymptotes of the hyperbola
form the diagonals of a rectangle defined by the major and minor axis; they intersect at the
hyperbola's center. Their slopes are thus $\pm \frac{a}{b} = \pm \frac{3}{5}$, so their equations in
point-slope form are $(y + 5) = \pm \frac{3}{5} \cdot (x+2)$. A graph of this curve is presented
in the answer key.
a) For this infinite series, \( \sum_{i=2}^{\infty} \frac{1}{i(i-1)} \), the most useful approach is to perform a decomposition by partial fractions and examine the result. We have

\[
\frac{1}{i(i-1)} = \frac{A}{i} + \frac{B}{i-1} = \frac{Ai - A + Bi}{i(i-1)} \Rightarrow A + B = 0 \Rightarrow B = 1.
\]

So we can write

\[
\sum_{i=2}^{\infty} \left( -\frac{1}{i} + \frac{1}{i-1} \right) = \left( -\frac{1}{2} + \frac{1}{1} \right) + \left( -\frac{1}{3} + \frac{1}{2} \right) + \ldots + \left( -\frac{1}{11} + \frac{1}{10} \right) + \left( -\frac{1}{12} + \frac{1}{11} \right)
\]

 cancels

\[
= 1 - \frac{1}{2} = \frac{1}{2}.
\]

b) The sum of an infinite geometric series is \( S = \frac{a}{1-r} \), where \( a \) is the initial term and \( r \) is the ratio between successive terms. To apply the formula accurately, it is important to be sure what the initial term is. For our series, the terms can be written out as

\[
\sum_{k=1}^{\infty} 5(-0.8)^k = 5(-0.8)^1 + 5(-0.8)^2 + 5(-0.8)^3 + \ldots
\]

So the ratio between terms is \( r = -0.8 \) and the first term is \( a = 5(-0.8) = -4 \).

The infinite sum is thus

\[
S = \frac{-4}{1-(-0.8)} = \frac{-4}{1.8} = \frac{-4}{18/10} = -\frac{40}{18} = -\frac{20}{9}.
\]

(13)

a) The general term of this series may be compared to a simpler rational function, thusly:

\[
\frac{n+5}{(n^2+n)^{\frac{3}{2}}} < \frac{n+5}{(n)^{\frac{3}{2}}} = \frac{n+5}{n^{\frac{3}{2}}} \quad \text{for } n > 1.
\]

So we have

\[
\sum_{n=1}^{\infty} \frac{n+5}{(n^2+n)^{\frac{3}{2}}} < \sum_{n=1}^{\infty} \frac{n+5}{n^{\frac{3}{2}}} = \sum_{n=1}^{\infty} \frac{n}{n^{\frac{3}{2}}} + \frac{5}{n^{\frac{3}{2}}} = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{2}}} + \frac{5}{n^{\frac{3}{2}}}.
\]

The general terms in the comparison series have the form \( \frac{1}{n^p} \); since \( p = \frac{1}{2} > 1 \) for the first term and \( p = \frac{3}{2} > 1 \) for the second, the comparison series converges by the "p-test". Therefore, our series converges by the Series Comparison Test.

b) If we examine the general term of this series, we find that \( \lim_{n \to \infty} \cos \left( \frac{\pi}{n} \right) = \cos 0 = 1 \).

This means that the terms in the infinite series tend to either -1 or +1, but not to zero.

So, by either the Theorem for Divergence or the Alternating Series Theorem, our series is divergent.
c) If we look at the progress of the coefficients of the terms in this power series, we see that
the factors in the numerator run 5, 8, 11, 14, ... and those in the denominator run 2, 6, 10, 14, ...
The first arithmetic sequence has a constant difference of 3. Since it begins at 5, but we wish

0.3.1 = 5 ⇒ 0 = 2. 
By a similar argument, the second sequence has a constant difference of 4 and begins at 2, so we have
b + 4n ⇒ b + 4.2 = 2 ⇒ b = -2.

Our power series can then be written as
∑_{n=1}^{∞} \frac{3n+2}{4n-2} x^n. 
The Ratio Test for

\lim_{n→∞} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n→∞} \left| \frac{(\frac{3(n+1)+2}{4(n+1)-2}) \cdot \frac{x^{n+1}}{x^n}}{(\frac{3n+2}{4n-2}) \cdot \frac{x^n}{x^n}} \right|

= \lim_{n→∞} \left| \frac{3n+5}{3n+2} \cdot \frac{4n-2}{4n+2} \cdot \frac{x}{x} \right| = |x| < 1.

So the radius of convergence of our power series is R = 1.

a) The general expression for a Taylor series expanded about a point x=a is

f(x) = f(a) + \frac{f'(a)}{1!} (x-a) + \frac{f''(a)}{2!} (x-a)^2 + \ldots + \frac{f^n(a)}{n!} (x-a)^n + \ldots = \sum_{n=0}^{∞} \frac{f^n(a)}{n!} (x-a)^n.

We wish to find a Taylor series for f(x) = \sin x, expanded about x = \frac{π}{6}.

The derivatives of sin x give us

f'(x) = \cos x ⇒ f'(\frac{π}{6}) = \cos \frac{π}{6} = \frac{\sqrt{3}}{2};
f''(x) = -\sin x ⇒ f''(\frac{π}{6}) = -\sin \frac{π}{6} = -\frac{1}{2};
f'''(x) = -\cos x ⇒ f'''(\frac{π}{6}) = -\cos \frac{π}{6} = -\frac{\sqrt{3}}{2};
f^{(4)}(x) = \sin x; \text{ etc. So our Taylor series looks like}

f(x) = \frac{1}{2} + \frac{\sqrt{3}}{1!} (x-\frac{π}{6}) - \frac{1}{2} \frac{\sqrt{3}}{1!} (x-\frac{π}{6})^2 - \frac{\sqrt{3}}{3!} (x-\frac{π}{6})^3 + \frac{1}{4!} (x-\frac{π}{6})^4 + \ldots ;
\text{ this can be regrouped and written as}

f(x) = \frac{1}{2} - \frac{\sqrt{3}}{2} \frac{(x-\frac{π}{6})^2}{2!} + \frac{1}{4} \frac{(x-\frac{π}{6})^4}{4!} + \ldots + \frac{\sqrt{3}}{3!} \frac{(x-\frac{π}{6})^3}{3!} - \frac{\sqrt{3}}{3!} \frac{(x-\frac{π}{6})^3}{3!} + \frac{\sqrt{3}}{3!} \frac{(x-\frac{π}{6})^3}{3!} - \ldots

= \frac{1}{2} \sum_{n=0}^{∞} \frac{(-1)^n}{(2n)!} (x-\frac{π}{6})^{2n} + \frac{\sqrt{3}}{2} \sum_{n=0}^{∞} \frac{(-1)^n}{(2n+1)!} (x-\frac{π}{6})^{2n+1}?
b) If we use a Taylor polynomial \( T_n(x) \), the difference or "remainder" from the correct value of \( f(x) \) is given by Taylor's Inequality:

\[
|R_n(x)| \leq \frac{M}{(n+1)!} |x-a|^{n+1},
\]

where \( M \) is the largest value of \( f^{(n+1)}(x) \) on the interval over which \( T_n(x) \) is being applied.

We would like to estimate \( \sin 29^\circ \) to a precision of \( 0.00001 = 10^{-5} \). We have constructed a Taylor series centered on \( 30^\circ = \frac{\pi}{6} \), so \( (x-a) = (29^\circ - 30^\circ) = -1^\circ = -\frac{\pi}{180} \) radian. All of the derivative of \( \sin x \) are either \( \sin x \) or \( \cos x \), so the largest value of any of these derivative in the neighborhood of \( a = \frac{\pi}{6} \) is about \( \frac{\sqrt{3}}{2} \); for calculation purposes, it will be quite adequate to set \( M = 1 \). We can guarantee a precision of \( 10^{-5} \) by solving the inequality

\[
\frac{M}{(n+1)!} \left| \frac{\pi}{180} \right|^{n+1} = \frac{1}{(n+1)!} \left( \frac{\pi}{180} \right)^{n+1} < 10^{-5} \quad \text{for } n.
\]

This is not algebraically straightforward, though, because of the \((n+1)!\) factor. Instead, we will need to estimate \( n \) by taking \( \frac{\pi}{180} \sim \frac{1}{60} \) and rearrange the inequality thus:

\[
10^{-5} \left( \frac{1}{60} \right)^{n+1} < (n+1)!
\]

Since \( \left( \frac{1}{60} \right)^2 = \frac{1}{3600} \), \( n=1 \) already gives us \( \frac{10^{-5}}{3600} = 27.8 \); this is not smaller than \( (1+1)! = 2! \), however. We do have a successful solution at \( n=2 \):

\[
10^{-5} \left( \frac{\pi}{180} \right)^3 < (3)! = 6.
\]

So using our series in part (a) only up to the second-degree term in \((x-a)\) should provide the desired precision. Thus, we may use

\[
T_2(x) = \frac{1}{2} + \frac{\sqrt{3}}{2} (x-\frac{\pi}{6}) - \frac{1}{4} (x-\frac{\pi}{6})^2,
\]

which uses the \( n=0 \) and \( n=1 \) terms from the first summation and \( n=0 \) from the second sum.

Only three terms are needed to estimate \( \sin 29^\circ \) to the desired precision of \( 10^{-5} \).

check: \( T_2(29^\circ) = \frac{1}{2} + \frac{\sqrt{3}}{2} (-\frac{\pi}{180}) - \frac{1}{4} (-\frac{\pi}{180})^2 \approx 0.5 - 0.013114595 - 0.000076158 = 0.484598888... 

by calculator - \sin 29^\circ \approx 0.484598888...


c) To produce a Maclaurin series for \( \sin x \), we will need to adopt our derivatives for \( a = 0 \).

Thus, \( f(a) = f'(a) = f''(a) = f'''(a) = ... = f^{(n)}(a) = 0 \) and \( f^{(2n+1)}(a) = (-1)^n \), so the series is \( \sin x = \sum_{n=0}^{\infty} (-1)^n x^{2n+1} \).

We can now go on to write

\[
\sin (2x) = \sum_{n=0}^{\infty} (-1)^n (2x)^{2n+1} = \sum_{n=0}^{\infty} (-1)^n \frac{(2x)^{2n+1}}{(2n+1)!}.
\]
a) Since we may write \[ \hat{a} \cdot \hat{b} = axb_x + ayb_y + azb_z = |\hat{a}| |\hat{b}| \cos \theta, \] the angle between 
\[ \hat{a} = \langle 8, -2, 3 \rangle \] and \[ \hat{b} = \langle -4, 5, -1 \rangle \] is given by 
\[
\cos \theta = \frac{\hat{a} \cdot \hat{b}}{|\hat{a}| |\hat{b}|} = \frac{(8)(-4) + (-2)(5) + (3)(-1)}{(\sqrt{8^2 + (-2)^2 + 3^2})(\sqrt{(-4)^2 + 5^2 + (-1)^2})} = \frac{-32 - 10 - 3}{(\sqrt{77})(\sqrt{42})} = \frac{-45}{\sqrt{77} \cdot \sqrt{42}}.
\]

With the aid of a calculator, we have \[ \theta = \cos^{-1}\left(\frac{-45}{\sqrt{77} \cdot \sqrt{42}}\right) \approx \cos^{-1}(0.75913) = 24.37 \text{ radians} \]
or \[ 143.3^\circ. \]

b) The magnitude of the projection of \(\hat{a}\) onto \(\hat{b}\) is given by 
\[
\cos \theta = \frac{\text{comp}_b \hat{a}}{|\hat{a}|} \Rightarrow \text{comp}_b \hat{a} = |\hat{a}| \cos \theta = \frac{|\hat{a}|}{|\hat{b}|} \cdot \left(\frac{\hat{a} \cdot \hat{b}}{|\hat{a}| |\hat{b}|}\right)
\]
\[
= \frac{\hat{a} \cdot \hat{b}}{|\hat{b}|} = \frac{\hat{a} \cdot \hat{b}}{|\hat{b}|^2} \hat{b} = \frac{-45}{42} \langle -4, 5, -1 \rangle = \frac{15}{14} \langle -4, 5, -1 \rangle.
\]

The projection vector itself has this magnitude and points in the direction of \(\hat{b}\). We can arrange this by multiplying a unit vector in \(\hat{b}\)'s direction by \(\text{comp}_b \hat{a}\). Hence, 
\[
\text{proj}_b \hat{a} = \left(\frac{\hat{a} \cdot \hat{b}}{|\hat{b}|^2}\right) \hat{b} = \frac{-45}{42} \langle -4, 5, -1 \rangle = \frac{15}{14} \langle -4, 5, -1 \rangle.
\]

values from part (a)

c) The area of the parallelogram formed by \(\hat{a}\) and \(\hat{b}\) is given by \[ |\hat{a}| |\hat{b}| \sin \theta. \] We could work this out from the information we already have. But this product is also the magnitude of the cross product of \(\hat{a}\) and \(\hat{b}\). The cross product can be calculated by using

\[
\hat{a} \times \hat{b} = \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
8 & -2 & 3 \\
-4 & 5 & -1
\end{vmatrix} = (2-15) \hat{i} - (-8+12) \hat{j} + (-40-8) \hat{k} = -13 \hat{i} - 4 \hat{j} + 32 \hat{k}.
\]

This is NOT a definition — it is only a convenient calculation method for three-dimensional vectors (ONLY!)

The magnitude of this vector, which also gives the area of the parallelogram is
\[
|\hat{a} \times \hat{b}| = \sqrt{(-13)^2 + (-4)^2 + 32^2} = \sqrt{1209} = 34.77.
\]

d) If a third vector \(\hat{c} = \langle 7, 0, -6 \rangle\) is used to define the sides of a parallelepiped built on a base of the parallelogram in part (c), the volume of this solid is given by
\[
(\hat{a} \times \hat{b}) \cdot \hat{c} = \begin{vmatrix}
8 & -2 & 3 \\
-4 & 5 & -1 \\
7 & 0 & -6
\end{vmatrix} = |(-30-0) \cdot 8 - (24+7) \cdot (-2) + (0-35) \cdot 3|
\]
\[
= |-240 + 62 - 105| = 183.
\]
6. There are a number of related ways to solve this problem; here is a straightforward one. If point C is on the same line as A and B, then the vector from C to A — call it \( \vec{a} \) — lies on the same line as the vector from C to B — call it \( \vec{b} \). Then \( \vec{b} \) is a scalar multiple of \( \vec{a} \), since \( \vec{a} \) and \( \vec{b} \) are either parallel or anti-parallel. We can work out this unknown scalar \( k \) from known components of the vectors, then proceed to find the unknown vector components.

First, we construct the vectors \( \vec{a} = \frac{CA}{CA} \) and \( \vec{b} = \frac{CB}{CB} \):

\[
\vec{a} = \begin{pmatrix} 3, -5, 2 \end{pmatrix} - \begin{pmatrix} -x, -1, 2 \end{pmatrix} = \begin{pmatrix} 3-x, -6, 0 \end{pmatrix},
\vec{b} = \begin{pmatrix} 4, 7, -3 \end{pmatrix} - \begin{pmatrix} -x, -1, 2 \end{pmatrix} = \begin{pmatrix} 4+x, 8, -5 \end{pmatrix}.
\]

We then find the scalar \( k \) such that \( \vec{b} = k \vec{a} \). We see that both \( k \) and \( \vec{b} \) are known, so \( b_x = k a_x \Rightarrow 4 = k(3-x) \Rightarrow k = \frac{4}{3-x} \). It is also the case that

\[
b_y = k a_y \Rightarrow 8 = k(-6) \Rightarrow k = \frac{8}{-6} = -\frac{4}{3},
\]

and

\[
b_z = k a_z \Rightarrow -5 = k(0) \Rightarrow k = 0.
\]

Thus, for \( C \) to be collinear, it must lie at \( \left( \frac{3}{2}, -1, \frac{5}{3} \right) \).

7. The lines \( L_1 \) and \( L_2 \) are given in parametric form,

\[
\begin{pmatrix} x-a \cr y-b \cr z-c \end{pmatrix} = \begin{pmatrix} s \cr t \end{pmatrix} \cdot \begin{pmatrix} a \cr b \cr c \end{pmatrix},
\]

which tells us that a line is parallel to the vector \( \langle a, b, c \rangle \). So first test that can be made is as to whether the two lines are parallel: if a second line is parallel to \( \langle a', b', c' \rangle \) and \( \langle a, b, c \rangle = k \langle a', b', c' \rangle \), where \( k \) is a scalar, then \( \langle a, b, c \rangle \) and \( \langle a', b', c' \rangle \) are parallel (or anti-parallel), and the two lines in question are also. We see for our lines that 

\[
L_1: \begin{pmatrix} x \cr y \cr z \end{pmatrix} = \begin{pmatrix} 1 \cr 4 \cr 2 \end{pmatrix} + \begin{pmatrix} 2 \cr 3 \cr 0 \end{pmatrix} s
\]

and

\[
L_2: \begin{pmatrix} x \cr y \cr z \end{pmatrix} = \begin{pmatrix} 1 \cr 4 \cr 2 \end{pmatrix} + \begin{pmatrix} -2 \cr -3 \cr 3 \end{pmatrix} t,
\]

we find that \( L_1 \) is parallel to \( \vec{v}_1 = \begin{pmatrix} 2 \cr 3 \cr 0 \end{pmatrix} \) and \( L_2 \) to \( \vec{v}_2 = \begin{pmatrix} -2 \cr -3 \cr 3 \end{pmatrix} \); but while \( s = \frac{3}{2} \) \( \vec{v}_1 \) and \( \vec{v}_2 = (-2) \vec{v}_1 \), we note that \( s = 3 \) \( \vec{v}_1 \) and \( -2 \vec{v}_2 = (-2) \vec{v}_1 \).

So \( L_1 \) and \( L_2 \) are not parallel.

The next test is to determine whether \( L_1 \) and \( L_2 \) intersect; if they do not, they are skew.

If we call the parameter for \( L_1 \), \( s \), and the parameter for \( L_2 \), \( t \), then we can write:

\[
\begin{align*}
L_1: & \quad (x-y, y-2, z-2) = \begin{pmatrix} 2 \cr 3 \cr 0 \end{pmatrix} s \\
x-y &= 2s \\
y-2 &= 3s \\
z-2 &= 0.
\end{align*}
\]

We can choose any pair of coordinates and solve for the values of \( s \) and \( t \) which give coordinates in common on the two lines. If these values of \( s \) and \( t \) give the same value for the remaining third coordinate, then \( L_1 \) and \( L_2 \) intersect; otherwise, they are skew.

Let us work with the \( y \)- and \( z \)-coordinates:

\[
\begin{align*}
L_1: & \quad \begin{pmatrix} y-7 \cr z-2 \end{pmatrix} = \begin{pmatrix} 3s \cr 0 \end{pmatrix} \Rightarrow \begin{pmatrix} y-7 \cr z-2 \end{pmatrix} = \begin{pmatrix} 3s \cr 0 \end{pmatrix} \\
y-7 &= 3s \\
z-2 &= 0.
\end{align*}
\]

Use first equation

Use second equation

If we now solve these equations for \( s \), \( y \), we find \( s = \frac{3}{2} \) \( \vec{v}_1 \) and \( \vec{v}_2 = \begin{pmatrix} -2 \cr -3 \cr 3 \end{pmatrix} \).

But also \( s = \frac{3}{2} \) \( \vec{v}_1 \) and \( \vec{v}_2 = \begin{pmatrix} -2 \cr -3 \cr 3 \end{pmatrix} \).

We have neglected the \( x \)-coordinate up until now. If \( x = -2 \) gives the same value of \( x \) for both \( L_1 \) as \( t = 9 \) does for \( L_2 \), then this is the point in common to both lines; otherwise, there is no intersection point. We have for \( L_1 \), \( x+4 = 2s = -2(2) = 4 \Rightarrow x = 0 \), but for \( L_2 \), \( x+1 = -4t = -4 \cdot -2 = 8 \Rightarrow x = 4 \).

Therefore, \( L_1 \) and \( L_2 \) have no point in common — they are skew lines.
We first need to find the line of intersection between the planes \( x - z = 1 \) and \( y + 2z = 3 \). Since this line is contained in both planes, it is perpendicular to the normal vectors of both planes. The normal vector to \( x - z = 1 \) is \( \vec{n}_1 = \langle 1, 0, -1 \rangle \) and the normal vector to \( y + 2z = 3 \) is \( \vec{n}_2 = \langle 0, 1, 2 \rangle \), so the line of intersection is parallel to

\[
\vec{n}_1 \times \vec{n}_2 = \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
1 & 0 & -1 \\
0 & 1 & 2 \\
\end{vmatrix} = (0+1)\hat{i} - (2-0)\hat{j} + (1-0)\hat{k} = \langle 1, -2, 1 \rangle = \vec{v}
\]

For the moment, this is all we need to know about this line.

The plane we are interested in is also perpendicular to the plane \( x + y - 2z = 1 \). So our plane also contains a vector parallel to the normal vector to that plane, \( \vec{n}_3 = \langle 1, 1, -2 \rangle \). If our plane contains both \( \langle 1, -2, 1 \rangle \) and \( \langle 1, 1, -2 \rangle \), we can construct a normal vector to it by taking the cross product

\[
\vec{v} \times \vec{n}_3 = \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
1 & -2 & 1 \\
1 & 1 & -2 \\
\end{vmatrix} = (4-1)\hat{i} - (-2-1)\hat{j} + (1+2)\hat{k} = \langle 3, 3, 3 \rangle.
\]

Now we need to know a point in our plane in order to describe it completely. Since the plane contains the intersection line between the planes \( x - z = 1 \) and \( y + 2z = 3 \), we can use any point which simultaneously satisfies \( x - z = 1 \Rightarrow z = x - 1 \) and \( y + 2z = 3 \Rightarrow z = \frac{y-3}{2} \). Since there are three coordinates, but only two equations (as we would expect since the set of solution points forms a line), so we may choose any value for \( z \) we wish, say, \( z = 0 \Rightarrow x = 1, y = 3 \). The point \( (1, 3, 0) \) is on the intersection line and thus in our plane. The equation for our plane is then

\[
3(x-1) + 3(y-3) + 3(z-0) = 0 \Rightarrow 3x + 3y + 3z = 12.
\]

(The above choice for \( z \) is not special — any other selection would lead to different values for \( x \) and \( y \) also, but would give us exactly the same equation for our plane.)
To transform this equation for a quadric surface in general form to an equation in standard form, we use a "completing the squares" method similar to that used on the conic section equation in Problem 11. We first group the terms involving \( x, y, \) or \( z \) to obtain

\[
\frac{x^2}{2} - \frac{x}{2} - \frac{y^2}{1} - \frac{y}{2} + \frac{z^2}{4} + \frac{x}{2} + \frac{z}{2} + 2 = 0.
\]

If we now extract common factors and add appropriate constants to "complete the squares", we find

\[
\begin{align*}
\left(\frac{x^2}{2} - \frac{x}{2} + \frac{1}{4}\right) - \left(\frac{y^2}{1} - \frac{y}{2} + \frac{1}{4}\right) + \left(\frac{z^2}{4} + \frac{z}{2} + \frac{1}{4}\right) &= -2 + \frac{1}{2} + \frac{1}{2} + \frac{1}{4} + \frac{1}{2} - \frac{1}{2} - \frac{1}{2} - 1.
\end{align*}
\]

\[
\Rightarrow \left(\frac{x}{1} - \frac{1}{2}\right)^2 - \left(\frac{y}{2} - \frac{1}{2}\right)^2 + \left(\frac{z}{2} + \frac{1}{2}\right)^2 = 2.
\]

All of the terms in the standard form are quadratic, so the surface is not a paraboloid or cone. Since there are negative terms, the surface is a hyperboloid; there is just one negative term, so this is a hyperboloid of one sheet. It is the "\( y \)-term" which is negative, so the "throat" of this hyperboloid is parallel to the \( y \)-axis. (We see this by setting \( x = 1, \ y = 1, \) or \( z = 2 \) in terms: \( x = 1 \) and \( z = 2 \) leave the equation of a hyperbola, while \( y = 1 \) leaves the equation of a circle. So the projections of the surface onto the \( xy \)- and \( xz \)-planes are hyperbolas, while the projection onto the \( xz \)-plane is a circle.) The center of our surface is located at \((1, 1, -2)\). A sketch of this figure is presented in the answer key.

2a) The equations of transformation between cylindrical and rectangular coordinates is

\[
x = r \cos \theta, \quad y = r \sin \theta, \quad z = z,
\]
so the reverse transformation is \( r^2 = x^2 + y^2 \), \( \theta = \tan^{-1}\left(\frac{y}{x}\right) \) (allowing for appropriate quadrant), and \( z = z \) (very like converting from rectangular to polar coordinates, except for the added dimension). So \((x, y, z) = (4, -4\sqrt{3}, -2)\) leads to \( r = \sqrt{x^2 + y^2} = \sqrt{16 + 48} = \sqrt{64} = 8 \) \( \theta = \tan^{-1}\left(-\frac{4\sqrt{3}}{4}\right) = \tan^{-1}\left(-\sqrt{3}\right) = -\frac{\pi}{3} \), \( z = -2 \). In cylindrical coordinates, we have \( P\left(8, \frac{\pi}{3}, -2\right) \) (quadrant IV).

b) In the label usage for spherical coordinates found in Stewart's book, \( \rho \) is the radial coordinate, \( \theta \) is the azimuthal coordinate, and \( \phi \) is the polar angle.

The transformation from spherical to rectangular coordinates is then

\[
x = \rho \sin \phi \cos \theta, \quad y = \rho \sin \phi \sin \theta, \quad z = \rho \cos \phi.
\]

For our point, \( \rho = 3, \ \theta = \frac{7\pi}{6}, \ \phi = \frac{3\pi}{4} \) yields \( x = 3 \sin \frac{3\pi}{4} \cos \frac{7\pi}{6} = 3 \left(\frac{\sqrt{2}}{2}\right) \left(-\frac{\sqrt{3}}{2}\right) = -\frac{3\sqrt{6}}{4}, \)
\[
y = 3 \sin \frac{3\pi}{4} \sin \frac{7\pi}{6} = 3 \left(\frac{\sqrt{2}}{2}\right) \left(-\frac{\sqrt{3}}{2}\right) = -\frac{3\sqrt{3}}{4}, \quad z = 3 \cos \frac{3\pi}{4} = 3 \left(-\frac{\sqrt{2}}{2}\right) = -\frac{3\sqrt{2}}{2}.
\]

So, in rectangular coordinates, we have \( P\left(-\frac{3\sqrt{6}}{4}, -\frac{3\sqrt{3}}{4}, -\frac{3\sqrt{2}}{2}\right) \).

[Please note that a different notation is widely used elsewhere for spherical coordinates, with \( \theta \) being the polar angle, \( \phi \) being the azimuthal angle, and the coordinates being presented in the order \((\rho, \theta, \phi)\). Also, "\( r \)" is frequently used instead of "\( \rho \).]