1) The calculation of this integral calls for integration by parts. In order that the new integral which will be produced be no more difficult than the one we started with, we choose \( u = x \) and \( dv = \cos x \, dx \). We then have

\[
\int_0^{\pi/4} x \cos x \, dx = \left[ x \sin x \right]_0^{\pi/4} - \left[ \int_0^{\pi/4} \sin x \, dx \right] = x \sin x \bigg|_0^{\pi/4} - (-\cos x) \bigg|_0^{\pi/4}
\]

\[
= x \sin x + \cos x \bigg|_0^{\pi/4} = \left( \frac{\pi}{4} \sin \frac{\pi}{4} + \cos \frac{\pi}{4} \right) - (0 \sin 0 + \cos 0)
\]

\[
= \left( \frac{\pi}{4} \cdot \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} \right) - (0 + 1) = \frac{\pi \sqrt{2}}{8} + \frac{\sqrt{2}}{2} - 1 \quad (B)
\]

2) This integral requires a certain “trick” in order to see how to integrate it. Starting from the Pythagorean Identity, \( \sin^2 x + \cos^2 x = 1 \), we divide through by \( \cos^2 x \), in order to obtain one of its alternative forms,

\[
\frac{\sin^2 x}{\cos^2 x} + \frac{\cos^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} \Rightarrow \tan^2 x + 1 = \sec^2 x.
\]

So our integral can be re-written as

\[
\int_0^{\pi/3} \tan^2 x \, dx = \int_0^{\pi/3} (\sec^2 x - 1) \, dx ;
\]

recalling that \( \frac{d}{dx} (\tan x) = \sec^2 x \) then lets us finish the job:

\[
\int_0^{\pi/3} (\sec^2 x - 1) \, dx = \tan x - x \bigg|_0^{\pi/3} = (\tan \frac{\pi}{3} - \frac{\pi}{3}) - (\tan 0 - 0)
\]

\[
= (\frac{\sin \frac{\pi}{3}}{\cos \frac{\pi}{3}} - \frac{\pi}{3}) - (0 - 0) = \left( \frac{\sqrt{3}}{2} - \frac{\pi}{3} \right) = \sqrt{3} - \frac{\pi}{3} \quad (E)
\]
3) For this trigonometric-powers integral, there are a few ways to approach it, but this is the most direct. We first separate out one factor of \( \cos x \) to obtain

\[
\int_0^{\pi/2} \sin^2 x \cos^3 x \, dx = \int_0^{\pi/2} \sin^2 x \cdot \cos x \, du.
\]

If we declare \( \cos x \, dx \) to be \( du \), then our substitution will be \( u = \sin x \). By applying the Pythagorean Identity, we can write \( \cos^2 x = 1 - \sin^2 x = 1 - u^2 \). We can now produce a substituted definite integral, using the substituted limits of integration shown in the table at right,

\[
\begin{array}{cccc}
 x: & 0 & \pi/2 \\
 u = \sin x: & 0 & 1 \\
\end{array}
\]

\[
\int_0^1 u^2 \cdot (1 - u^2) \, du = \int_0^1 u^2 - u^4 \, du = \left[ \frac{1}{3} u^3 - \frac{1}{5} u^5 \right]_0^1 = \left( \frac{1}{3} \cdot 1^3 - \frac{1}{5} \cdot 1^5 \right) - \left( \frac{1}{3} \cdot 0^3 - \frac{1}{5} \cdot 0^5 \right) = \frac{1}{3} - \frac{1}{5} = \frac{5}{15} = \frac{2}{15} \quad (D)
\]

4) In order to decide on a decomposition for the method of partial fractions, we should look at the denominator of the rational function. This contains a repeated linear factor, \( (3x + 2)^2 \), and an irreducible quadratic factor, since \( x^2 + 2 \) has no real zeroes. (Upon inspecting this rational function, we see that the factors in the denominator would produce a fourth-degree polynomial (leading term: \( 9x^4 \)), while the numerator is a third-degree polynomial (leading term: \( 15x^3 \)). So no polynomial division is needed here first.)

For a repeated linear factor, the decomposition must contain terms having denominators with every power of that factor up to the power appearing in the rational function. As these are linear factor terms, the numerators must be constants. So there need to be two terms:

\[
\frac{C_1}{3x + 2} + \frac{C_2}{(3x + 2)^2}.
\]

Since the quadratic term cannot be factored further, we use it as it stands; the numerator for a quadratic factor term is a linear factor. This term in the decomposition must then be

\[
\frac{C_3 x + C_4}{x^2 + 2}.
\]

The complete partial fraction decomposition can thus be written as

\[
\frac{15 x^3 + 33 x^2 + 14 x + 4}{(3x + 2)^2 (x^2 + 2)} = \frac{Ax + B}{x^2 + 2} + \frac{C}{(3x + 2)^2} + \frac{D}{(3x + 2)} \quad (C)
\]
5) For this integral, we see that a “u-substitution” will work: if we choose \( u = 1 - x^2 \) for the denominator, the rest of the integral can be written in terms of \( du \), giving us

\[
\int_{4/5}^{3/5} \frac{x}{1 - x^2} \, dx \quad u = 1 - x^2 \implies \frac{du}{dx} = -2x \implies -\frac{1}{2}du = x \, dx
\]

\[
x : \quad \frac{4}{5} \quad \frac{3}{5}
\]

\[
\int_{9/25}^{16/25} -\frac{1}{2} \frac{du}{u} \quad -\frac{1}{2} \ln \left| u \right| \bigg|_{9/25}^{16/25} = -\frac{1}{2} \left( \ln \frac{16}{25} - \ln \frac{9}{25} \right)
\]

\[
= -\frac{1}{2} \ln \left( \frac{16}{25} \right) = -\frac{1}{2} \ln \frac{16}{9} = \frac{1}{2} \ln \frac{9}{16} = \ln \left( \frac{3}{4} \right)^{1/2} = \ln \left( \frac{3}{4} \right) \quad (B)
\]

\[
\log a - \log b = \log \left( \frac{a}{b} \right) \quad -\log a = \log \left( \frac{1}{a} \right) \quad p \log a = p \log a^p
\]

6) The first thing to notice about this integrand is that it looks something like the derivative of the arctangent function:

\[
\frac{d}{dx} \tan^{-1} x = \frac{1}{1 + x^2}.
\]

So we are going to need to make some manipulation of the integrand to bring it into that form:

\[
\int_{0}^{3/2} \frac{1}{9 + 4x^2} \, dx = \int_{0}^{3/2} \frac{1}{9} \cdot \frac{1}{1 + \left( \frac{2}{3} x \right)^2} \, dx = \frac{1}{9} \int_{0}^{3/2} \frac{1}{1 + \left( \frac{2}{3} x \right)^2} \, dx.
\]

We can now make a u-substitution:

\[
\frac{1}{9} \cdot \int_{0}^{3/2} \frac{1}{1 + \left( \frac{2}{3} x \right)^2} \, dx \quad u = \frac{2}{3} x \implies du = \frac{2}{3} \, dx \implies \frac{3}{2} du = dx
\]

\[
x : \quad 0 \quad \frac{3}{2}
\]

\[
\int_{0}^{1} \frac{1}{1 + u^2} \left( \frac{3}{2} \, du \right) = \frac{1}{9} \cdot \frac{3}{2} \int_{0}^{1} \frac{1}{1 + u^2} \, du = \frac{1}{6} \cdot \tan^{-1} x \bigg|_{0}^{1}
\]

\[
= \frac{1}{6} \left( \tan^{-1} 1 - \tan^{-1} 0 \right) = \frac{1}{6} \left( \frac{\pi}{4} - 0 \right) = \frac{\pi}{24} \quad (C)
\]
7) The infinitesimal element of arclength in the (Euclidean) plane is \( ds = \sqrt{dx^2 + dy^2} \), so the arclength along a curve from point A to point B is always found from

\[
 s = \int_A^B ds = \int_A^B \sqrt{dx^2 + dy^2} .
\]

Since we will be integrating along the \( x \)-direction, we need to factor out the differential \( dx \), making our arclength integral

\[
 s = \int_A^B \sqrt{\frac{dx^2}{dx^2} + \frac{dy^2}{dx^2}} \, dx = \int_1^2 \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx .
\]

We now need to find the derivative \( dy/dx \):

\[
y = \ln x \implies \frac{dy}{dx} = \frac{1}{x}
\]

in order to complete the expression for the arclength integral

\[
 s = \int_1^2 \sqrt{1 + \left(\frac{1}{x}\right)^2} \, dx = \int_1^2 \sqrt{1 + \frac{1}{x^2}} \, dx = \int_1^2 \sqrt{\frac{x^2 + 1}{x^2}} \, dx = \int_1^2 \frac{\sqrt{x^2 + 1}}{x} \, dx
\]

(A)

8) The differential equation \( xy' = -y \), or \( x \cdot \frac{dy}{dx} = -y \), is separable as \( \frac{dy}{y} = -\frac{dx}{x} \).

Upon integrating both sides of this equation, we find

\[
 \int \frac{1}{y} \, dy = \int -\frac{1}{x} \, dx \implies \ln|y| = -\ln|x| + C.
\]

We can then exponentiate both sides to obtain

\[
e^{\ln|y|} = e^{-\ln|x|+C} \implies |y| = e^{-\ln|x|} \cdot e^C = A \cdot e^{\ln|\frac{1}{x}|} = A \cdot \frac{1}{x}.
\]

(exponentiation makes the additive constant, \( C \), a multiplicative constant, \( A \); \(-\log x = \log \left(\frac{1}{x}\right)\); since we are given \( x > 0 \), the absolute value sign can be dropped)

The general solution for the differential equation is thus \( y = A/x \), for \( x > 0 \). Since we are told that \( y(5) = 2 \), we have

\[
y = 2 = \frac{A}{5} \implies A = 10,
\]

making the solution to the initial-value problem \( y = 10/x \). (A)
The surface area integral for a solid of revolution can always be written as
\[ A = \int 2\pi \cdot r \, ds , \]
where \( r \) is the perpendicular distance from the axis of revolution to the curve being revolved and \( ds \) is the infinitesimal element of arclength along the curve. For this problem, the axis of revolution is the \( x \)-axis, so the perpendicular distance will be \( r = y \).

The curve being revolved is parametric with respect to \( t \), so we will need to factor \( dt \) out of the arclength element. So our surface area integral is
\[
A = \int 2\pi \cdot r \, ds = \int 2\pi \cdot y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt
\]
\[
= \int_0^1 2\pi \cdot y(t) \cdot \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt .
\]

We need the derivatives for the parametric curve in order to complete the description of this integral:
\[
x = \frac{2}{3} t^{3/2} + t \quad \Rightarrow \quad \frac{dx}{dt} = \frac{2}{3} \cdot \frac{3}{2} \cdot t^{1/2} + 1 = t^{1/2} + 1 ,
\]
\[
y = \frac{2}{3} t^{3/2} - t \quad \Rightarrow \quad \frac{dy}{dt} = t^{1/2} - 1 .
\]

\[
\Rightarrow \quad A = \int_0^1 2\pi \cdot \left(\frac{2}{3} t^{3/2} - t\right) \cdot \sqrt{(t^{1/2} + 1)^2 + (t^{1/2} - 1)^2} \, dt
\]
\[
= 2\pi \cdot \int_0^1 \left(\frac{2}{3} t^{3/2} - t\right) \cdot \sqrt{t + 2t^{1/2} + 1 + (t - 2t^{1/2} + 1)} \, dt
\]
\[
= 2\pi \cdot \int_0^1 \left(\frac{2}{3} t^{3/2} - t\right) \cdot \sqrt{2t + 2} \, dt
\]
\[
= 2\pi \cdot \sqrt{2} \cdot \int_0^1 \left(\frac{2}{3} t^{3/2} - t\right) \cdot \sqrt{t + 1} \, dt . \quad (A)
\]
10) The transformation equations for converting rectangular (Cartesian) coordinates to polar coordinates give us

\[ r = \sqrt{x^2 + y^2} = \sqrt{(-5\sqrt{3})^2 + 15^2} = \sqrt{25 \cdot 3 + 225} = \sqrt{300} = 10\sqrt{3} \]

and

\[ \tan \theta = \frac{y}{x} = \frac{15}{-5\sqrt{3}} = -\frac{3}{\sqrt{3}} = -\sqrt{3} \cdot \]

The arctangent function can only produce angles in the fourth and first quadrants \((-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2})\), giving us \(\tan^{-1}(-\sqrt{3}) = -\frac{\pi}{3}\). Since our point is in the second quadrant, with \(x < 0\) and \(y > 0\), we must use \(\theta = \pi + (-\frac{\pi}{3}) = \frac{2\pi}{3}\). Hence, the polar coordinates for our point are \((10\sqrt{3}, \frac{2\pi}{3})\). (C)

11) A useful convergence test for infinite series such as this one is the “\(p\)-test”, which is a consequence of the Integral Test for convergence:

the series \(\sum_{n=1}^{\infty} \frac{1}{n^p}\) converges for \(p > 1\).

Our series can be written as \(\sum_{n=1}^{\infty} \frac{n^\alpha + 1}{n^4} = \sum_{n=1}^{\infty} \frac{n^\alpha}{n^4} + \sum_{n=1}^{\infty} \frac{1}{n^4}\). For the series in the second term, \(p = 4\), so it converges. The series in the first term can be viewed as

\[ \sum_{n=1}^{\infty} \frac{n^\alpha}{n^4} = \sum_{n=1}^{\infty} n^{\alpha - 4} = \sum_{n=1}^{\infty} \frac{1}{n^{-4}} \cdot \frac{1}{n^\alpha} = \sum_{n=1}^{\infty} \frac{1}{n^{4-\alpha}}, \]

which will converge only for \(p = 4 - \alpha > 1 \Rightarrow 4 - 1 = 3 > \alpha\). Our original series converges when the series in both terms converge, which is when \(\alpha < 3\). (C)
The cross product of two vectors produces a vector which is perpendicular to both of those vectors. For the vectors \( \mathbf{u} = <1, 0, 3> \) and \( \mathbf{v} = <2, 3, 0> \), we have

\[
\mathbf{u} \times \mathbf{v} = \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
1 & 0 & 3 \\
2 & 3 & 0
\end{vmatrix}
= \hat{i} \begin{vmatrix} 3 & 1 \\ 0 & 2 \end{vmatrix} - \hat{j} \begin{vmatrix} 0 & 1 \\ 3 & 2 \end{vmatrix} + \hat{k} \begin{vmatrix} 0 & 3 \\ 1 & 2 \end{vmatrix}
= \hat{i}(-9) - \hat{j}(-6) + \hat{k}(3)
= -9\hat{i} - 6\hat{j} + 3\hat{k}
= \langle -9, 6, 3 \rangle.
\]

* The 3 x 3 determinant provides a means of calculating the cross product of two three-dimensional vectors, but is not a definition.

Any scalar multiple of \( \langle -9, 6, 3 \rangle \) is also orthogonal to \( \mathbf{u} \) and \( \mathbf{v} \); among the available choices, we find \( \frac{1}{5}\langle -9, 6, 3 \rangle = \langle 3, -2, -1 \rangle \). (C)

We can check this by calculating the dot product of this new vector with each of \( \mathbf{u} \) and \( \mathbf{v} \):

\[
\langle 3, -2, -1 \rangle \cdot \langle 1, 0, 3 \rangle = (3)(1) + (-2)(0) + (-1)(3) = 3 + 0 - 3 = 0,
\]

\[
\langle 3, -2, -1 \rangle \cdot \langle 2, 3, 0 \rangle = (3)(2) + (-2)(3) + (-1)(0) = 6 - 6 + 0 = 0.
\]

Both dot products are zero, so the new vector is perpendicular to both \( \mathbf{u} \) and \( \mathbf{v} \).

The distance from a plane to a point not located in that place is measured along a line through the point which is perpendicular to the plane. A normal vector to the plane \( x + 2y + 3z = 4 \) is \( <1, 2, 3> \) (and this plane clearly does not contain the origin, \( (0, 0, 0) \)). So the symmetric equations for a line which passes through the origin and is parallel to \( <1, 2, 3> \), and is thus perpendicular to the plane, are

\[
x - 0 \over 1 = y - 0 \over 2 = z - 0 \over 3 \quad \text{or} \quad x = \frac{y}{2} = \frac{z}{3}.
\]

If we solve this set of equations to express the variables, say, in terms of \( x \), we have \( y = 2x \) and \( z = 3x \). Using these in the equation for the plane gives us

\[
x + 2 \cdot (2x) + 3 \cdot (3x) = x + 4x + 9x = 14x = 4 \quad \Rightarrow \quad x = \frac{2}{7} \quad \Rightarrow \quad y = \frac{4}{7}, \quad z = \frac{6}{7}.
\]

This is the intersection point on the plane with the normal line which passes through the origin; it is thus the closest point in the plane to the origin. The distance of this point from the origin is

\[
\sqrt{\left(\frac{2}{7}\right)^2 + \left(\frac{4}{7}\right)^2 + \left(\frac{6}{7}\right)^2} = \sqrt{\frac{4}{49} + \frac{16}{49} + \frac{36}{49}} = \sqrt{\frac{56}{49}} = \frac{8}{7}.
\]

To bring this result into line with the choices available, we can multiply the numerator and denominator by \( \sqrt{2} \):

\[
\frac{8}{7} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{16}{\sqrt{14}} = \frac{4}{\sqrt{14}}. \quad (D)
\]
14) To bring the given equation into the standard form for a sphere, we will need to do some algebraic manipulation and “completion of squares”:

\[-\frac{2}{3} x + x^2 + (-2 + y)^2 + 8y - 2z + z^2 = \frac{26}{9}\]

\[\Rightarrow x^2 - \frac{2}{3} x + y^2 - 4y + 4 + 8y + z^2 - 2z = \frac{26}{9}\]

\[\Rightarrow \left( x^2 - \frac{2}{3} x + \frac{1}{9} \right) + \left( y^2 + 4y + 4 \right) + \left( z^2 - 2z + 1 \right) = \frac{26}{9} + \frac{1}{9} + 1 \]

\[\left( \frac{1}{3} \cdot \frac{2}{3} \right)^2 = 1/9 \quad \left( \frac{1}{2} \cdot 1 \right)^2 = 1\]

\[\Rightarrow \left( x - \frac{1}{3} \right)^2 + \left( y + 2 \right)^2 + \left( z - 1 \right)^2 = \frac{36}{9} = 4 \quad . \]

With the equation now in standard form \((x - h)^2 + (y - k)^2 + (z - l)^2 = r^2\), we can immediately read off that the sphere is centered at \((h, k, l) = (1/3, -2, 1)\) and has radius \(r = 2\). \(\text{(B)}\)

15) In the convention used in our textbook (Stewart) for spherical coordinates, \(\theta\) is the “azimuthal” angle measured along the equator of the sphere counter-clockwise from the positive x-axis and \(\phi\) is the “polar” angle measured from the upper or “north” pole of the sphere. In this system, the equations for the transformation from spherical coordinates \((r, \theta, \phi)\) to rectangular coordinates \((x, y, z)\) are

\[x = r \cos \theta \sin \phi \quad , \quad y = r \sin \theta \sin \phi \quad , \quad z = r \cos \phi \quad . \]

For the point \((r, \theta, \phi) = (4, \pi/6, \pi/3)\), we thus find

\[(x, y, z) = \left( 4 \cos \frac{\pi}{6} \sin \frac{\pi}{3} \right) \quad , \quad \left( 4 \sin \frac{\pi}{6} \sin \frac{\pi}{3} \right) \quad , \quad \left( 4 \cos \frac{\pi}{3} \right) \]

\[= \left( 4 \cdot \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2} \right) \quad , \quad \left( 4 \cdot \frac{\sqrt{3}}{2} \cdot \frac{1}{2} \right) \quad , \quad \left( 4 \cdot \frac{1}{2} \right) \]

\[= \left( 3, \sqrt{3}, 2 \right) \quad . \quad \text{(A)}\]

Be aware that other labeling conventions exist: in many other books and in other fields, such as physics, \(\theta\) denotes the “polar angle” and \(\phi\), the “azimuthal angle”, using the coordinate order \((r, \theta, \phi)\).
16) We can test this series for convergence using the Integral Test:

\[ \sum_{n=1}^{\infty} n \cdot e^{-n^2} \to \int_{1}^{\infty} xe^{-x^2} \, dx \quad u = -x^2 \Rightarrow du = -2x \, dx \Rightarrow -\frac{1}{2} \, du = x \, dx \]

\[
x: \quad 1 \quad \infty
\]

\[
u = -x^2: \quad -1 \quad -\infty
\]

\[= \int_{-1}^{-\infty} e^{u} \cdot (-\frac{1}{2} \, du) = \frac{1}{2} \cdot \int_{-\infty}^{-1} e^{u} \, du = \frac{1}{2} \cdot (e^{-1} - e^{-\infty}) = \frac{1}{2} \cdot (e^{-1} - 0) = \frac{1}{2e},
\]

which is finite, so the series converges;

or by the Ratio Test:

\[
\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{(n+1) \cdot e^{-(n+1)^2}}{n \cdot e^{-n^2}} \right| = \lim_{n \to \infty} \left| \frac{(n+1)}{n} \cdot e^{-n^2 - 2n - 1 - (-n^2)} \right|
\]

\[= \lim_{n \to \infty} \left| \frac{(n+1)}{n} \cdot e^{-2n+1} \right| = 1 \cdot e^{-\infty} = 1 \cdot 0 = 0 < 1 ,
\]

so the series converges.

17)

a) For a parametric curve, the derivative at a point given by \((x(t), y(t))\) is

\[
\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} .
\]

For our curve, the derivatives are

\[
x = t^3 - 3t^2 \quad \Rightarrow \quad \frac{dx}{dt} = 3t^2 - 6t \quad \text{and} \quad y = t^3 - 3t \quad \Rightarrow \quad \frac{dy}{dt} = 3t^2 - 3 .
\]

Thus, we obtain

\[
\frac{dy}{dx} = \frac{3t^2 - 3}{3t^2 - 6t} = \frac{3 \cdot (t^2 - 1)}{3 \cdot (t^2 - 2t)} = \frac{(t - 1) \cdot (t + 1)}{t \cdot (t - 2)} .
\]

b) The tangent line to this curve is horizontal at any point where \(\frac{dy}{dx} = 0\). For our derivative, this will be the case where the numerator is zero, provided the denominator is not also zero there; this occurs at \(t = -1\) and \(t = +1\). The points with horizontal tangents are then

\[
(x(-1), y(-1)) = ([-1]^3 - 3 \cdot [-1]^2, [-1]^3 - 3 \cdot [-1]) = (-1 - 3, -1 + 3) = (-4, 2) \quad \text{and} \quad (x(+1), y(+1)) = (1^3 - 3 \cdot 1^2, 1^3 - 3 \cdot 1) = (1 - 3, 1 - 3) = (-2, -2) .
\]

(continued)
c) The tangent line to this curve is vertical at any point where \( \frac{dy}{dx} \) is undefined. For our derivative, this will be the case where the denominator is zero, provided the numerator is not also zero there; this occurs at \( t = 0 \) and \( t = +2 \). The points with vertical tangents are then

\[
(x(0), y(0)) = (0^3 - 3 \cdot 0^2, 0^3 - 3 \cdot 0) = (0 - 0, 0 - 0) = (0,0) \quad \text{and} \\
(x(+2), y(+2)) = (2^3 - 3 \cdot 2^2, 2^3 - 3 \cdot 2) = (8 - 12, 8 - 6) = (-4,2).
\]

Notice that there is something peculiar about the point \((-4, 2)\): it has both a horizontal and a vertical tangent. This is generally an indication that the curve has crossed itself at such a point; a graph will be important in investigating the actual circumstances.

d, e) The curve rises (falls) over those intervals of \( t \) for which \( \frac{dy}{dx} > 0 \) (\( \frac{dy}{dx} < 0 \)). In looking at the rational function

\[
\frac{dy}{dx} = \frac{3t^2 - 3}{3t^2 - 6t} = \frac{(t - 1)(t + 1)}{t \cdot (t - 2)},
\]

we found that the critical points occur at the parameter values \( t = -1, 0, +1, +2 \). A full-blown analysis of the slope function in a rational inequality would require eight cases, so it will be considerably simpler to just look at the signs of the factors and determine the sign of the slope with the intervals among the critical points:

<table>
<thead>
<tr>
<th>( t )</th>
<th>((-\infty, -1))</th>
<th>((-1, 0))</th>
<th>((0, 1))</th>
<th>((1, 2))</th>
<th>((2, \infty))</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{dy}{dx} )</td>
<td>-</td>
<td>-</td>
<td>+</td>
<td>+</td>
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<td>+</td>
</tr>
</tbody>
</table>

Thus, this parametric curve is rising for the intervals of \( t: (-\infty, -1), (0, 1), \) and \((2, \infty)\) and falling for \( t: (-1, 0)\) and \((1, 2)\).

f) A graph for this curve is sketched in the answer key.
The general term for this power series is \( a_n = \frac{(x-1)^n}{2^n \cdot (n^2 + 1)} \). The radius of convergence for the series is found by application of the Ratio Test:

\[
\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{(x-1)^{n+1}}{2^{n+1} \cdot ([n+1]^2 + 1)} \cdot \frac{2^n \cdot (n^2 + 1)}{(x-1)^n} \right| = \lim_{n \to \infty} \left| \frac{(x-1)^{n+1}}{(x-1)^n} \cdot \frac{2^n}{2^{n+1}} \cdot \frac{n^2 + 1}{n^2 + 2n + 2} \right|
\]

\[
= \left| (x-1) \cdot \frac{1}{2} \cdot 1 \right| < 1 \implies |x-1| < 2 .
\]

So the radius of convergence of our series is \( R = 2 \) (and the interval of convergence is centered on \( x = 1 \)). To determine completely the interval of convergence, we must examine the behavior of the series at each endpoint:

\[
x = 3: \sum_{n=1}^{\infty} \frac{(3-1)^n}{2^n \cdot (n^2 + 1)} = \sum_{n=1}^{\infty} \frac{2^n}{2^n \cdot (n^2 + 1)} = \sum_{n=1}^{\infty} \frac{1}{(n^2 + 1)},
\]

which is convergent (proof: \( \frac{1}{n^2 + 1} < \frac{1}{n^2} \); since \( \sum_{n=1}^{\infty} \frac{1}{n^2} \) converges by the “p-test” (Integral Test), then \( \sum_{n=1}^{\infty} \frac{1}{(n^2 + 1)} \) also converges (Comparison Tests);

\[
x = -1: \sum_{n=1}^{\infty} \frac{([-1]-1)^n}{2^n \cdot (n^2 + 1)} = \sum_{n=1}^{\infty} \frac{(-2)^n}{2^n \cdot (n^2 + 1)} = \sum_{n=1}^{\infty} \frac{(-1)^n \cdot 2^n}{2^n \cdot (n^2 + 1)} = \sum_{n=1}^{\infty} \frac{(-1)^n}{(n^2 + 1)},
\]

since we have already shown that \( \sum_{n=1}^{\infty} \frac{1}{(n^2 + 1)} \) is absolutely convergent, this alternating series certainly converges. Hence, the interval of convergence for our series is \([ -1 , 3 ]\).
In order to obtain a precision of \( 0.0001 = 10^{-4} \) using a Taylor polynomial, the first term not included from the full infinite series must be no larger than this desired precision. Taylor’s Inequality states that the “error” or remainder produced by using only the terms up to index \( n \) is no larger than the next term (the first unused term) in the Taylor series:

\[
|R_n(x)| \leq \frac{M}{(n+1)!} |x-a|^{n+1} \leq 10^{-4}, \text{ with } |f^{(n+1)}(x)| \leq M,
\]

\( M \) being a bound on the size of the \((n+1)^{th}\) derivative of \( f(x) \) in the interval where the Taylor polynomial is being applied.

We want to estimate \( \cos 42^\circ \) to a precision of \( 10^{-4} \). The closest value for which we know a value of \( \cos x \) is \( a = 45^\circ = \pi/4 \); this makes our interval span \( 3^\circ \) to either side of 45º, or \( 3^\circ \cdot (\pi \text{ radians} / 180^\circ) = \pi/60 \) radians. The higher derivatives of our function \( f(x) = \cos x \) all involve either \( \sin x \) or \( \cos x \). The bound \( M \) we would properly use in Taylor’s Inequality is the number with the largest absolute value for \( \sin x \) or \( \cos x \) in the interval \( |x-\pi/4| \leq \pi/60 \). Most people find this excessively fussy: it will not affect our result significantly if we simply use the largest absolute value that \( \sin x \) or \( \cos x \) can ever have, so we will just set \( M = 1 \).

This at last leaves us to solve the inequality

\[
\frac{M}{(n+1)!} |x-a|^{n+1} = \frac{1}{(n+1)!} \cdot \left( \frac{\pi}{60} \right)^{n+1} \leq 10^{-4}
\]

for the smallest value of \( n \) which will make it true. Unfortunately, because of the presence of the factorial factor \((n+1)!\), this cannot be solved algebraically, so we will have to try values of \( n \) to see what works. We can make a good guess at what the solution is by using the fact that \( \pi/60 \approx 3/60 = 1/20 \), so \( (1/20)^2 = 1/400 \) and \( (1/20)^3 = 1/8000 \). We want a product that is less than \( 1/10,000 \), so we find that

\[
\begin{align*}
n+1 &= 2: \quad \frac{1}{(n+1)!} \cdot \left( \frac{\pi}{60} \right)^{n+1} \approx \frac{1}{2!} \cdot \left( \frac{1}{20} \right)^2 = \frac{1}{2} \cdot \frac{1}{400} = \frac{1}{800}, \\
n+1 &= 3: \quad \frac{1}{(n+1)!} \cdot \left( \frac{\pi}{60} \right)^{n+1} \approx \frac{1}{3!} \cdot \left( \frac{1}{20} \right)^3 = \frac{1}{6} \cdot \frac{1}{8000} = \frac{1}{48,000}.
\end{align*}
\]

We thus need to take the Taylor polynomial out to \( (n+1) = 3 \) terms, or

\[
T_2(x) = \cos \left( \frac{\pi}{4} \right) + \frac{-\sin \left( \frac{\pi}{4} \right)}{1!} \cdot (x - \frac{\pi}{4}) + \frac{-\cos \left( \frac{\pi}{4} \right)}{2!} \cdot (x - \frac{\pi}{4})^2
\]

to obtain four decimal places of precision for \( \cos 42^\circ \), working from \( a = 45^\circ \).

(continued)
We are not asked to *make* the actual calculation, but if we were to do so, we would obtain

\[
\cos 42^\circ \approx T_2(42^\circ) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} \cdot (-\frac{\pi}{60}) + \frac{\sqrt{2}}{2} \cdot (-\frac{\pi}{60})^2
\]

\[
\approx 0.707107 + 0.037024 - 0.000969 \approx 0.743162.
\]

The result from a calculator (which uses a longer Taylor polynomial in order to display nine decimal places of precision) is \( \cos 42^\circ \approx 0.743145 \), so we have indeed reached the desired precision of 10^-4 or better. The next term in the Taylor series would be

\[
\frac{\sin(\frac{\pi}{4})}{3!} \cdot (x - \frac{\pi}{4})^3 = \frac{\sqrt{2}}{6} \cdot (-\frac{\pi}{60})^3 \approx 0.0000169.
\]

20) A parallelepiped is a three-dimensional figure with six faces, where the faces opposite each other are identical parallelograms. For three vectors, not all in the same plane, which extend from a single point, two of the vectors can be chosen to define a base parallelogram and the third vector will then establish the volume. The order in which these vectors are chosen turns out to make no difference in finding the volume, which we obtain by taking the absolute value of a scalar triple product of the three vectors,

\[
V = \left| \vec{a} \cdot (\vec{b} \times \vec{c}) \right|.
\]

For the diagram given, the three vectors emanate from the point A. We will label them

\[
\vec{b} = \langle 0, 3, 2 \rangle - \langle -1, 2, 0 \rangle = \langle 1, 1, 2 \rangle, \\
\vec{c} = \langle -3, 0, 3 \rangle - \langle -1, 2, 0 \rangle = \langle -2, -2, 3 \rangle, \quad \text{and} \quad \\
\vec{d} = \langle -2, 1, 1 \rangle - \langle -1, 2, 0 \rangle = \langle -1, -1, 1 \rangle.
\]

It doesn’t matter for calculating the volume of the parallelepiped in what order we take the vectors in the scalar triple product. We will choose, say, \( \vec{c} \cdot (\vec{b} \times \vec{d}) \); the scalar triple product can be found using the determinant aid

\[
\vec{c} \cdot (\vec{b} \times \vec{d}) \quad \text{"} \* \text{"} \quad \begin{vmatrix} c_x & c_y & c_z \\ b_x & b_y & b_z \\ d_x & d_y & d_z \end{vmatrix} = \begin{vmatrix} -2 & -2 & 3 \\ 1 & 1 & 2 \\ -1 & -1 & 1 \end{vmatrix}
\]

*see remark in Problem 12

\[
= (\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix}) - (\begin{vmatrix} 1 & 2 \\ -1 & -1 \end{vmatrix}) + (3) \begin{vmatrix} 1 & 1 \\ -1 & -1 \end{vmatrix} = (-2)(3) + (2)(3) + (3)(0) = 0.
\]

Those who are more familiar with determinants might recognize that this determinant is zero because two of its columns are identical.

The volume of this figure turns out to be zero, meaning that all three vectors are, in fact, lying in one plane (or, we say, the three vectors are coplanar).

G. Ruffa – 12/08