1) This fraction appears in Problem 5 of the “undated-2002?” exam; a solution can be found in that solution set. (E)

2) This integral appears in Problem 6 of the Fall 2004 exam; a solution can be found in that solution set. (C)

3) This integral appears in Problem 1 of the Fall 2001 exam; a solution can be found in that solution set. (B)

4) This integral appears in Problem 2 of the Fall 2001 exam; a solution can be found in that solution set. (E)

5) This curve appears in Problem 12 of the “undated-2002?” exam; a solution can be found in that solution set. (C)

6) The surface area for a solid of revolution is given by the integral \( A = \int 2\pi \cdot r \, ds \), where \( r \) is the distance from the axis of rotation to a point on the curve being revolved and \( ds \) is the infinitesimal element of arclength at that point along the curve. Since the rotation axis in this problem is the x-axis, the radial distance is \( r = y \).
The element of arc length is always \( ds = \sqrt{dx^2 + dy^2} \). We are dealing with a parametric curve, so we will need to factor out the differential \( dt \), giving us

\[
A = 2\pi \cdot \int_0^{\pi/4} y(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt
\]

We can then work out the derivatives for the parametric functions,

\[
\frac{d}{dt} x(t) = \frac{d}{dt} \left( t^2 \sin 2t \right) = 2t \sin 2t + 2t^2 \cos 2t,
\]

\[
\frac{d}{dt} y(t) = \frac{d}{dt} \left( t^2 \cos 2t \right) = 2t \cos 2t - 2t^2 \sin 2t,
\]

in order to complete the surface area integral

\[
A = 2\pi \cdot \int_0^{\pi/4} \left( t^2 \cos 2t \right) \sqrt{\left( 2t \sin 2t + 2t^2 \cos 2t \right)^2 + \left( 2t \cos 2t - 2t^2 \sin 2t \right)^2} \, dt
\]

\[
= 2\pi \cdot \int_0^{\pi/4} t^2 \cos 2t \cdot \sqrt{4t^2 \left( \sin^2 2t + \cos^2 2t \right)^2 + \left( \cos 2t - t \sin 2t \right)^2} \, dt
\]

\[
= 4\pi \cdot \int_0^{\pi/4} t^3 \cos 2t \cdot \sqrt{\sin^2 2t + 2t \sin 2t \cos 2t + t^2 \cos^2 2t} + \left( \cos^2 2t - 2t \cos 2t \sin 2t + t^2 \sin^2 2t \right) \, dt
\]

\[
= 4\pi \cdot \int_0^{\pi/4} t^3 \cos 2t \cdot \sqrt{\sin^2 2t + \cos^2 2t} + t^2 \left( \cos^2 2t + \sin^2 2t \right) \, dt
\]

\[
= 4\pi \cdot \int_0^{\pi/4} t^3 \cos 2t \cdot \sqrt{1 + t^2} \, dt \quad \text{(A)}
\]

7) This point appears in Problem 10 of the Fall 2001 exam; a solution can be found in that solution set. \( \text{(C)} \)

8) This function appears in Problem 13 of the “undated-2002?” exam; a solution can be found in that solution set. \( \text{(C)} \)
9) It will be easiest to decide among the choices for this question by looking at the
conditions for convergence of the original series, \( \sum_{n=1}^{\infty} \frac{(-1)^n}{(n^2+1)^p} \), and its absolute series,
\( \sum_{n=1}^{\infty} \frac{1}{(n^2+1)^p} \). We will use as a comparison series \( \sum_{n=1}^{\infty} \frac{1}{(n^2)^p} = \sum_{n=1}^{\infty} \frac{1}{n^{2p}} \).

By the “p-test”, a corollary of the Integral Test, we know that the series \( \sum_{n=1}^{\infty} \frac{1}{n^{2p}} \) converges for \( 2p > 1 \). Since \( \frac{1}{(n^2+1)^p} < \frac{1}{n^{2p}} \), the absolute series also converges for \( 2p > 1 \); thus, we can say that the original series is absolutely convergent for \( p > \frac{1}{2} \). The original series is an alternating series, for which the general term is \( b_n = \frac{1}{(n^2+1)^p} \).

According to the Alternating Series Test, the original series converges if \( b_n \geq b_{n+1} \) and \( \lim_{n \to \infty} b_n = 0 \); both of these conditions are met for \( p > 0 \). In summary then, we can say that the original series is

- absolutely convergent for \( p > \frac{1}{2} \),
- conditionally convergent for \( 0 < p \leq \frac{1}{2} \), and
- divergent for \( p \leq 0 \).

Among the available choices, the only one with a correct statement of the convergence conditions is \( \text{(E)} \).

10) A parallelepiped is a three-dimensional figure with six faces, in which opposite faces are identical parallelograms. If three vectors are chosen to emanate from a single point, any two can be used to construct a “base” parallelogram, for which the third vector can then be used to produce the remaining faces in three dimensions. The volume of this solid is found from the absolute value of the “scalar triple product” of the three vectors, \( \vec{a} \cdot (\vec{b} \times \vec{c}) \), with the vectors being labeled in any order.

If we choose to identify the vectors as \( \vec{a} = <1, 1, 0> \), \( \vec{b} = <1, 0, 1> \), and \( \vec{c} = <0, 1, 1> \), then the scalar triple product is given by

\[
\begin{vmatrix}
1 & 1 & 0 \\
0 & 1 & 1 \\
1 & 0 & 1 \\
\end{vmatrix}
= 1 \cdot 0 \cdot 1 - 1 \cdot 0 \cdot 1 + 0 \cdot 1 \cdot 1
= \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix}
= 1 \cdot 1 - 1 \cdot 0 - 0 \cdot 1 + 1 \cdot 0
= 1 - 0 - 0 + 0
= 1.
\]

* the 3 x 3 determinant is an aid to computing the triple product, but is not a definition for it

The absolute value of this product, which gives the volume of the parallelepiped, is 2. \( \text{(B)} \)

11) This series appears in Problem 10 of the “undated-2002?” exam; a solution can be found in that solution set. \( \text{(E)} \)
12) The differential equation \( xy' = -y \), or \( x \cdot \frac{dy}{dx} = -y \), is separable as \( \frac{dy}{y} = \frac{-dx}{x} \).

Upon integrating both sides of this equation, we find

\[
\int \frac{1}{y} \, dy = \int \frac{-1}{x} \, dx \quad \Rightarrow \quad \ln|y| = -\ln|x| + C.
\]

We can then exponentiate both sides to obtain

\[
e^{\ln|y|} = e^{-\ln|x| + C} \quad \Rightarrow \quad |y| = e^{-\ln|x|} \cdot e^C = A \cdot e^{\ln|x|/x} = A \cdot \frac{1}{x}.
\]

exponentiation makes the additive constant, C, a multiplicative constant, A; \(-\log x = \log (1/x);\) since we are given \( x > 0 \), the absolute value sign can be dropped.

The general solution for the differential equation is thus \( y = A/x \), for \( x > 0 \). Since we are told that \( y(5) = 2 \), we have

\[
y = 2 = \frac{A}{5} \quad \Rightarrow \quad A = 10,
\]

making the solution to the initial-value problem \( y = 10/x \). Consequently, we find the value for the function, \( y(1) = 10/1 = 10 \). \( \text{(A)} \)

13) We will investigate each of these series in turn.

\[
\sum_{n=1}^{\infty} \frac{\sin(1/n)}{1/n} : \quad \text{If we consider the limit of the general term of this series, after}
\]

making the variable transformation \( m = 1/n \), we see that

\[
\lim_{n \to \infty} \frac{\sin(1/n)}{1/n} = \lim_{m \to 0} \frac{\sin m}{m} = 1.
\]

By the Divergence Test, an infinite series in which the limit of the general term is not zero diverges. Hence, this series is divergent.

\[
\sum_{n=1}^{\infty} \left( \frac{\sin 1}{n} \right)^2 : \quad \text{Since } 0 < 1 < \pi/2, \text{ we have } 0 < \sin 1 < 1; \text{ thus it is so that}
\]

\[
\left( \frac{\sin 1}{n} \right)^2 < \frac{1}{n^2}.
\]

By the “p-test”, \( \sum_{n=1}^{\infty} \frac{1}{n^2} \) converges \( (p = 2 > 1) \).

So, by the Comparison Tests, our series is also convergent.

(continued)
\[
\sum_{n=1}^{\infty} \sin^2\left(\frac{1}{n}\right): \text{ We will compare the general term of this series, } a_n = \sin^2\left(\frac{1}{n}\right), \text{ to the term } b_n = \frac{1}{n^2}; \text{ for the first series above, we found that }
\lim_{n \to \infty} \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} = 1; \text{ it will therefore be true that }
\lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{\sin^2\left(\frac{1}{n}\right)}{\frac{1}{n^2}} = 1^2 = 1; \text{ we know that the series } \sum_{n=1}^{\infty} \frac{1}{n^2} \text{ converges by the "p-test" ( } p = 2 > 1 \text{ ); by the Limit Comparison Test, then, our series also converges.}
\]

14) This integral is not in a suitable form to solve immediately, as there is no factor we can include in the differential to facilitate a substitution (and integration by parts will not be at all helpful). Since the natural exponential function presents itself, however, we could start by multiplying both numerator and denominator by \( e^x \):

\[
\int_{-\infty}^{\infty} \frac{1}{e^x + e^{-x}} \, dx = \int_{-\infty}^{\infty} \frac{1}{e^x + e^{-x}} \cdot \frac{e^x}{e^x} \, dx = \int_{-\infty}^{\infty} \frac{e^x}{e^{2x} + 1} \, dx.
\]

Now, the substitution \( u = e^x \) will transform the integral into one we recognize at once:

\[
\int_{-\infty}^{\infty} \frac{e^x}{e^{2x} + 1} \, dx \quad u = e^x \Rightarrow du = e^x \, dx \quad \text{integration limits are unchanged}
\]

\[
= \int_{-\infty}^{\infty} \frac{1}{u^2 + 1} \, du = \lim_{t \to \infty} \tan^{-1} u \bigg|_{-t}^{t} = \lim_{t \to \infty} \left( \tan^{-1} t - \tan^{-1} [-t] \right)
\]

\[
= \frac{\pi}{2} - (-\frac{\pi}{2}) = \pi.
\]

15) This is another integral without an immediately obvious substitution: the term \( x^8 \) has no simple relation through differentiation to the factor \( x^4 \). But that factor of \( x^4 \) suggests, in turn, the use of a substitution based on \( x^4 \), with \( x^3 \) used in the differential. We will then have

\[
\int x^3 \sqrt{1 - x^8} \, dx \quad u = x^4 \Rightarrow du = 4x^3 \, dx \Rightarrow \frac{1}{4} du = x^3 \, dx
\]

\[
\Rightarrow x^8 = u^2
\]

(continued)
\[
\int \sqrt{1-u^2} \cdot \frac{1}{4} \, du = \frac{1}{4} \int \sqrt{1-u^2} \, du.
\]

It is clear now that the original integral will require a combination of methods.
This transformed integral must be solved by a trigonometric substitution:

\[
\frac{1}{4} \int \sqrt{1-u^2} \, du \quad \sin \theta = \frac{u}{1} = u \Rightarrow du = \cos \theta \, d\theta
\]

\[
\Rightarrow \cos \theta = \frac{\sqrt{1-u^2}}{1} = \sqrt{1-u^2}
\]

\[
= \frac{1}{4} \int \cos \theta \cdot \cos \theta \, d\theta
\]

we now employ the trigonometric identity of the “double-angle formula” for cosine:

\[
\cos 2\theta = \cos^2 \theta - \sin^2 \theta = \cos^2 \theta - (1 - \cos^2 \theta) \Rightarrow \cos^2 \theta = \frac{1 + \cos 2\theta}{2},
\]

in order to complete the solution of this integral:

\[
\frac{1}{4} \int \cos^2 \theta \, d\theta = \frac{1}{4} \int \frac{1}{2} \cdot (1 + \cos 2\theta) \, d\theta = \frac{1}{8} \int (1 + \cos 2\theta) \, d\theta
\]

\[
= \frac{1}{8} \left( \theta + \frac{1}{2} \sin 2\theta \right) = \frac{1}{8} \theta + \frac{1}{16} \cdot (2 \sin \theta \cos \theta)
\]

\[
= \frac{1}{8} \left( \sin^{-1} u + u \cdot \sqrt{1-u^2} \right) + C = \frac{1}{8} \left( \sin^{-1}(x^4) + x^4 \sqrt{1-x^8} \right) + C
\]

16)

\[ r = 1 + 2 \sin \theta \]

a) It is useful here to plot first a graph of the function \( r(\theta) = 1 + 2 \sin \theta \) versus \( \theta \).
We observe that a portion of the curve shows a negative value for the radius of the curve: this corresponds to the “inner loop” that this limaçon makes in the first and second quadrants. We will be interested in this loop in part (c). A plot of the polar curve itself is presented in the Answer Key.

(continued)
b) The infinitesimal element of arclength in the (Euclidean) plane is \( ds = \sqrt{dx^2 + dy^2} \) (see also Problem 6 above), so the arclength of a curve between points A and B is given by \( s = \int_A^B ds = \int_A^B \sqrt{dx^2 + dy^2} \). A polar curve is a type of parametric curve, so we will wish to factor out the differential \( d\theta \), in order to compute the arclength around the complete limaçon,

\[
s = \int_{\text{full period}} \sqrt{dx^2 + dy^2} = \int_0^{2\pi} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} \, d\theta = \int_0^{2\pi} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} \, d\theta = \int_0^{2\pi} \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2} \, d\theta \, .
\]

Because we are working with a polar curve, we need to express the Cartesian differentials \( dx \) and \( dy \) in terms of the polar variables \( r \) and \( \theta \). This further requires that we keep in mind that \( r \) is a function of \( \theta \), so finding the derivatives of \( x \) and \( y \) entails the use of the Product Rule:

\[
x = r \cos \theta \quad \Rightarrow \quad \frac{dx}{d\theta} = \frac{dr}{d\theta} \cos \theta + r \cdot \frac{d}{d\theta} \cos \theta = \frac{dr}{d\theta} \cos \theta - r \sin \theta \, ,
\]

\[
y = r \sin \theta \quad \Rightarrow \quad \frac{dy}{d\theta} = \frac{dr}{d\theta} \sin \theta + r \cdot \frac{d}{d\theta} \sin \theta = \frac{dr}{d\theta} \sin \theta + r \cos \theta \, .
\]

This simplifies our arclength integral expression to

\[
s = \int_0^{2\pi} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} \, d\theta = \int_0^{2\pi} \sqrt{\left(\frac{dr}{d\theta} \cos \theta - r \sin \theta\right)^2 + \left(\frac{dr}{d\theta} \sin \theta + r \cos \theta\right)^2} \, d\theta
\]

\[
= \int_0^{2\pi} \sqrt{\left[\left(\frac{dr}{d\theta}\right)^2 \cos^2 \theta - 2 \cdot \frac{dr}{d\theta} \cos \theta \cdot \sin \theta + r^2 \sin^2 \theta\right] + \left[\left(\frac{dr}{d\theta}\right)^2 \sin^2 \theta + 2 \cdot \frac{dr}{d\theta} \sin \theta \cdot \cos \theta + r^2 \cos^2 \theta\right]} \, d\theta
\]

\[
= \int_0^{2\pi} \sqrt{\left(\frac{dr}{d\theta}\right)^2 \left(\cos^2 \theta + \sin^2 \theta\right) + r^2 \cdot \left(\sin^2 \theta + \cos^2 \theta\right)} \, d\theta = \int_0^{2\pi} \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2} \, d\theta.
\]

We now need only find the derivative \( \frac{dr}{d\theta} (r(\theta)) \), in order to set up our arclength integral:

\[
\frac{dr}{d\theta} = \frac{d}{d\theta} (1 + 2 \sin \theta) = 2 \cos \theta \quad \Rightarrow
\]

\[
s = \int_0^{2\pi} \sqrt{(2 \cos \theta)^2 + (1 + 2 \sin \theta)^2} \, d\theta = \int_0^{2\pi} \sqrt{4 \cos^2 \theta + (1 + 4 \sin \theta + 4 \sin^2 \theta)} \, d\theta
\]

\[
= \int_0^{2\pi} \sqrt{5 + 4 \sin \theta} \, d\theta,
\]

which is always defined, since \(|4 \sin \theta| < 5\).

As this is a fairly challenging integral to evaluate, you are not asked to do it on the exam.
c) The area within a polar curve \( r(\theta) \) between angles \( \theta_1 \) and \( \theta_2 \) is the sum of infinitesimal triangular wedges radiating from the origin to the curve, given by

\[
A = \int_{\theta_1}^{\theta_2} \frac{1}{2} [r(\theta)]^2 \, d\theta .
\]

In order to evaluate the area of the “inner loop”, we will need to determine the range of angles for the integration. This loop is formed by the part of the limaçon where the radius is negative, so we wish to find the angles for which \( r(\theta) = 0 \):

\[
r(\theta) = 1 + 2 \sin \theta = 0 \quad \Rightarrow \quad \sin \theta = -\frac{1}{2} \quad \Rightarrow \quad \theta = \frac{7\pi}{6}, \frac{11\pi}{6} .
\]

Thus, our polar area integral is

\[
A = \int_{\frac{7\pi}{6}}^{\frac{11\pi}{6}} \left( 1 + 4 \sin \theta + 4 \sin^2 \theta \right) d\theta .
\]

using the “double-angle formula” for cosine, we find \( \cos 2\theta = \cos^2 \theta - \sin^2 \theta = (1 - \sin^2 \theta) - \sin^2 \theta \Rightarrow \sin^2 \theta = \frac{1 - \cos 2\theta}{2} \), which allows us to re-write the integral as

\[
A = \frac{1}{2} \int_{\frac{7\pi}{6}}^{\frac{11\pi}{6}} \left( 1 + 4 \sin \theta + 4 \sin^2 \theta \right) d\theta = \frac{1}{2} \int_{\frac{7\pi}{6}}^{\frac{11\pi}{6}} \left( 1 + 4 \sin \theta + 4 \cdot \left[ \frac{1 - \cos 2\theta}{2} \right] \right) d\theta
\]

\[
= \frac{1}{2} \int_{\frac{7\pi}{6}}^{\frac{11\pi}{6}} (1 + 4 \sin \theta + 2 - 2 \cos 2\theta) d\theta = \frac{1}{2} \int_{\frac{7\pi}{6}}^{\frac{11\pi}{6}} (3 + 4 \sin \theta - 2 \cos 2\theta) d\theta
\]

\[
= \frac{1}{2} \left( 3\theta - 4 \cos \theta - \sin 2\theta \right) \bigg|_{\frac{7\pi}{6}}^{\frac{11\pi}{6}}
\]

\[
= \frac{1}{2} \left[ \left( \frac{33\pi}{6} - 4 \cos \frac{11\pi}{6} - \sin \frac{22\pi}{6} \right) - \left( \frac{21\pi}{6} - 4 \cos \frac{7\pi}{6} - \sin \frac{14\pi}{6} \right) \right]
\]

\[
= \frac{1}{2} \left( \frac{33\pi}{6} - 4 \cos \frac{11\pi}{6} - \sin \frac{5\pi}{3} - \frac{21\pi}{6} + 4 \cos \frac{7\pi}{6} + \sin \frac{\pi}{3} \right)
\]

\[
= \frac{1}{2} \left( \frac{12\pi}{6} - 4 \cdot \frac{\sqrt{3}}{2} - \left[ -\frac{\sqrt{3}}{2} \right] + 4 \cdot \left[ -\frac{\sqrt{3}}{2} \right] + \frac{\sqrt{3}}{2} \right)
\]

\[
= \frac{1}{2} \left( 2\pi - 6 \cdot \frac{\sqrt{3}}{2} \right) = \pi - \frac{3\sqrt{3}}{2} \approx 0.5435 .
\]

(continued)
We can compare the inner loop of the limaçon to a circle of diameter 1 and area \( \frac{1}{4} \pi d^2 = \frac{1}{4} \pi \approx 0.7854 \) to see that our result is credible.

17)

a) We can develop the equation for the desired plane once we know a normal vector for it. What we know about this plane is the equation for a line it contains and the position of a point in the plane which is not on that line.

We will be able to obtain the normal vector as the vector cross product of two vectors in the plane. In turn, we can construct these vectors by choosing two points on the line in the plane and extending vectors from them which pass through the specified point. (We know that the point \((1, 2, 3)\) does not lie on the line, as there is no single value of the parameter \(t\) in the equation for the line which works for all three coordinates.)

We may choose values of the parameter \(t\) that make computation easy. We will set \(t = 0\) for point A and \(t = 1\) for point B:

\[
A : (3 \cdot 0, 1 + 0, 2 - 0) = (0, 1, 2), \quad B : (3 \cdot 1, 1 + 1, 2 - 1) = (3, 2, 1).
\]

The vectors from the external point C \((1, 2, 3)\) to these points on the line are

\[
CA = \langle 0 - 1, 1 - 2, 2 - 3 \rangle = \langle -1, -1, -1 \rangle \quad \text{and} \quad CB = \langle 3 - 1, 2 - 2, 1 - 3 \rangle = \langle 2, 0, -2 \rangle.
\]

A normal vector to the plane containing these vectors is then found from the vector cross product

\[
\begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
-1 & -1 & -1 \\
2 & 0 & -2
\end{vmatrix}
= \begin{vmatrix}
-1 & -1 \\
0 & -2
\end{vmatrix} \hat{i} - \begin{vmatrix}
-1 & -1 \\
2 & -2
\end{vmatrix} \hat{j} + \begin{vmatrix}
-1 & -1 \\
2 & 0
\end{vmatrix} \hat{k}
\]

\[
= (2 - 0) \hat{i} - (2 - [-2]) \hat{j} + (0 - [-2]) \hat{k} = <2, -4, 2>.
\]
We now know the coefficients for the equation of the plane; we can then use the coordinates of point C, for instance, to determine the complete equation
\[2 \cdot (x - 1) - 4 \cdot (y - 2) + 2 \cdot (z - 3) = 0\]
\[\Rightarrow 2x - 2 - 4y + 8 + 2z - 6 = 0 \Rightarrow 2x - 4y + 2z = 0.\]

We can try the coordinates of points A, B, or C in this equation to see that it indeed contains these points.

b) The set of symmetric equations for the line normal to our plane and passing through point C (1, 2, 3) is then directly created from the components of the normal vector:
\[
\frac{x - 1}{2} = \frac{y - 2}{4} = \frac{z - 3}{2}.
\]

18a)

a) The general term for this power series is
\[a_n = \frac{(-1)^n \cdot x^{2n+2}}{(2n+1)(2n+2) \cdot 3^{2n+1}}.
\]
The radius of convergence for the series is found by application of the Ratio Test:
\[
\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{(-1)^{n+1} \cdot x^{2(n+1)+2}}{(2n+3)(2n+4) \cdot 3^{2(n+1)+1}} \right|
\]
\[
= \lim_{n \to \infty} \left| \frac{(-1)^n \cdot x^{2n+4}}{(-1)^n \cdot x^{2n+2} \cdot (2n+3)(2n+4) \cdot 3^{2n+3}} \right|
\]
\[
= \left| -1 \cdot x^2 \cdot 1 \cdot 1 \cdot \frac{1}{3^2} \right| = \left| x^2 \right| < 1 \Rightarrow \left| x \right| < 9 \Rightarrow \left| x \right| < 3.
\]
So the radius of convergence of our series is \( R = 3 \).

b) The derivative of a sum of terms is equal to the sum of the derivatives of the terms; this is no less true if the sum is an infinite series. So we may find the derivative of the function represented by our series, \( f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^{2n+2}}{(2n+1)(2n+2) \cdot 3^{2n+1}} \), by differentiating the general term (which is sometimes referred to as “differentiating under the summation sign”):

(continued)
\[ f'(x) = \frac{d}{dx} \left( \sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^{2n+2}}{(2n+1)(2n+2) \cdot 3^{2n+1}} \right) = \sum_{n=0}^{\infty} \frac{d}{dx} \left( \frac{(-1)^n \cdot x^{2n+2}}{(2n+1)(2n+2) \cdot 3^{2n+1}} \right) \]
\[ = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(2n+2) \cdot 3^{2n+1}} \cdot \frac{d}{dx} (x^{2n+2}) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(2n+2) \cdot 3^{2n+1}} \cdot (2n+2) \cdot x^{2n+1} \]
\[ = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)} \cdot \left( \frac{x}{3} \right)^{2n+1}. \]

We are asked to recognize this alternating series as the Maclaurin series for the arctangent function, with the one modification that the variable the function acts on is \( x/3 \). Thus, we have that \( f'(x) = \tan^{-1}(x/3) \); the series we have just found describes correctly the behavior of this derivative function for \(|x| < 3\). So we can reasonably evaluate the series for \( x = \sqrt{3} \) by calculating
\[ f'(\sqrt{3}) = \tan^{-1}(\frac{\sqrt{3}}{3}) = \frac{\pi}{6}. \]

G. Ruffa – 1/09