1. The linear function described would have the form $\log_6 y = m(\log_6 x) + b$; this is to say that on a "log-log" plot, the graph would be a straight line with slope $m$ and intercept $b$. We are told that the function assumes the value $\log_6 y = 1$ where $\log_6 x = 0$, so $\log_6 y = 1 = m \cdot 0 + b \Rightarrow b = 1$.

If we now exponentiate both sides of this equation with a base of 10, we find

$$10^{\log_6 y} = 10^{m(\log_6 x) + 1}$$

$$\Rightarrow y = 10^m \cdot 10^{\log_6 x} \cdot 10$$

$$= (10^{\log_6 x})^m \cdot 10$$

$$= x^m \cdot 10$$

So the function is $y = 10 \cdot x^m$, $m$ being the slope on the log-log plot (the exam problem uses "b").

2. For the area bounded by $y = x$ and $y = 2 + 2x - x^2$, we may perform the integration once we have found the points of intersection of the curves, which we define the limits of integration. If we set the two functions equal, we find

$$x = 2 + 2x - x^2 \Rightarrow x^2 - x - 2 = 0$$

$$\Rightarrow (x + 1)(x - 2) = 0.$$ 

So the intersections occur at $x = -1$ and $x = 2$.

The area integral is then

$$\int_{-1}^{2} (2 + 2x - x^2) - x \, dx = \left[ -x^2 + x + 2 \right]_{-1}^{2}$$

$$= -\frac{1}{3} (2)^3 + \frac{1}{2} (2)^2 + 2 \cdot 2 + \frac{1}{3} (-1)^3 - \frac{1}{2} (-1)^2 - 2 (-1)$$

$$= -\frac{8}{3} + 2 + 4 - \frac{1}{3} - \frac{1}{2} + 2 = 8 - 3\frac{1}{2} = \frac{9}{2}$$

3. We may find the derivative of $f(x) = x^{g(x)}$ by logarithmic differentiation:

$$y = x^{g(x)} \Rightarrow \ln y = g(x) \cdot \ln x \Rightarrow \frac{dy}{dx} = g'(x) \cdot \ln x + g(x) \cdot \frac{1}{x}.$$ 

Thus $f'(x) = \left[ g'(x) \cdot \ln x + \frac{g(x)}{x} \right] \cdot x^{g(x)}$

$$f'(2) = \left[ g'(2) \cdot \ln 2 + \frac{3(2)}{2} \right] \cdot 2^{g(2)} = \left[ 4 \cdot \ln 2 + 3 \cdot 2 \right] \cdot 2^3$$

$$= 8 \cdot \ln 2 + 8 \cdot \frac{3}{2} = 32 \ln 2 + 12$$
a) This limit produces the indeterminate difference \( \infty - \infty \); it will be of help to place the terms on a common denominator:

\[
\frac{1}{\sin x} - \frac{2}{1 - \cos x} = \frac{1}{\sin x} \cdot \frac{\sin x}{\sin x} - \frac{2}{1 - \cos x} \cdot \frac{1 + \cos x}{1 + \cos x} = \frac{\sin x}{\sin^2 x} - \frac{2(1 + \cos x)}{\sin^2 x} = \frac{\sin x - 2 - 2 \cos x}{\sin^2 x}
\]

The limit \( \lim_{x \to 0} \frac{\sin x - 2 - 2 \cos x}{\sin^2 x} = \frac{-4}{0} = -\infty \) is no longer indeterminate.

b) The sine function is a bounded function, since \(-1 \leq \sin x \leq 1\); we then see that \( e^{-1} \leq e^{\sin x} \leq e^{1} \), since \( e^x \) and \( \sin x \) are continuous, so the inequality will be properly behaved. For positive real values of \( x \), then, \( \frac{e^{-1}}{x} \leq \frac{e^{\sin x}}{x} \leq \frac{e^{1}}{x} \). We can conclude from the "Squeeze" or "Sandwich" Theorem that because \( \lim_{x \to 0} \frac{e^{-1}}{x} = 0 \) and \( \lim_{x \to 0} \frac{e^{1}}{x} = 0 \), then \( \lim_{x \to \infty} \frac{e^{\sin x}}{x} = 0 \).

c) If we simply follow the "Limit Laws", we find that \( \lim_{x \to 0} \frac{e^{2x} - 2x - 1}{x^2 - x^3} = \frac{e^0 - 2 \cdot 0 - 1}{0^2 - 0^3} = \frac{0}{0} \);

this is an indeterminate ratio suitable for evaluation by the use of L'Hôpital's Rule:

\[
\lim_{x \to 0} \frac{e^{2x} - 2x - 1}{x^2 - x^3} = \lim_{x \to 0} \frac{(e^{2x} - 2x - 1)'}{(x^2 - x^3)'} = \lim_{x \to 0} \frac{2e^{2x} - 2}{2x - 3x^2} = \frac{2e^0 - 2}{2 \cdot 0 - 3 \cdot 0^2} = \frac{0}{0} \r
\]

we apply the Rule once more:

\[
\lim_{x \to 0} \frac{2e^{2x} - 2}{2x - 3x^2} = \lim_{x \to 0} \frac{(2e^{2x} - 2)'}{(2x - 3x^2)'} = \lim_{x \to 0} \frac{4e^{2x}}{2 - 6x} = \frac{4 \cdot e^0}{2 - 6 \cdot 0} = \frac{4}{2} = 2
\]

d) There's no trick here: we can simply apply the Limit Laws—

\[
\lim_{x \to 0} \frac{x^2 + 2}{x + 3} = \frac{0 + 2}{0 + 3} = \frac{2}{3}
\]

\(5\) We will apply the definition \( f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \) to the function \( f(x) = \frac{3}{x} \), \( x \neq 0 \).

\[
f'(x) = \lim_{h \to 0} \frac{\frac{3}{x+h} - \frac{3}{x}}{h} = \lim_{h \to 0} \frac{\frac{3x - 3(x+h)}{x(x+h)} h}{h} = \lim_{h \to 0} \frac{3x - 3x - 3h}{h \cdot x(x+h)} = \lim_{h \to 0} \frac{-3h}{h \cdot x(x+h)} = \lim_{h \to 0} \frac{-3}{x(x+h)} = \frac{-3}{x^2}
\]
a) A function $f(x)$ is increasing wherever $f'(x) > 0$ and decreasing where $f'(x) < 0$. This can be read directly from the graph of $f(x)$. So, for our function, $f(x)$ is increasing on $[-3,-2)$ and $(0,2]$ and is decreasing on $(-2,0]$.

b) A function $f(x)$ has upward concavity wherever $f''(x) > 0$ and downward concavity where $f''(x) < 0$. This must be found from the graph by looking at the slope of the curve for $f(x)$, since $f''(x) = \frac{d}{dx}[f'(x)]$. For our function, then, $f(x)$ is concave upward on $(-1,1)$ and $(1,2)$ and is concave downward on $(-3,-1)$.

c) Since $f(x)$ is finite on this interval, the slope of $f(x)$ is always defined, so $f(x)$ has no vertical asymptotes on this interval.

d) We see from the graph of $f'(x)$ that $f(x)$ has a local maximum at $x = -2$ (if $f(-2) = 0$, $f''(-2) < 0$), a local minimum at $x = 0$ (if $f(0) = 0$, $f''(0) > 0$) and points of inflection at $x = -1$ and $x = 1$ (if $f''(x) = 0$ and $f'(x) \neq 0$). A graph of $f(x)$ is shown in the answer key.

e) We have set $f(0) = 0$ already. The "Net Change Theorem" tells us that $\int_{a}^{b} f(x) \, dx = F(b) - F(a)$, where $F'(x) = f(x)$ for $f(x)$ continuous on $(a,b)$. Since $f'(x) < 0$ on $(-2,0)$, the integral $\int_{-2}^{0} f(x) \, dx$ will be negative, so $f(0) - f(-2) < 0$. The area between the $x$-axis and the curve for $f(x)$ is larger than a triangle with a base of 2 and a height of 1 in the interval $[-2,0]$, so the signed area is less than 1 (its magnitude is larger than 1). So $\int_{-2}^{0} f(x) \, dx = f(0) - f(-2) < -1 \Rightarrow f(-2) > f(0) + 1 = 0 + 1 = 1$. 


a) In a recursive process such as this, an equilibrium point is a value of $x$ for which the function $f(x)$ returns the same value. To find these points for our function, we set

$$f(x) = \frac{5}{8} x (4-x) = x \Rightarrow \frac{5}{8} x^2 - 4 \cdot \frac{5}{8} x + x = \frac{5}{8} x^2 - \frac{3}{2} x = 0$$

$$\Rightarrow x \left( \frac{5}{8} x - \frac{3}{2} \right) = 0 \Rightarrow \text{either } x = 0 \text{ or } \frac{5}{8} x - \frac{3}{2} = 0$$

$$\Rightarrow x = \frac{8}{5} \cdot \frac{3}{2} = \frac{12}{8}$$

The equilibria lie at $x = 0$ and $x = \frac{12}{8} = 2.4$.

b) An equilibrium point $x = a$ is stable (an attractor) if a change in the value of $x_A$ away from $a$ produces an opposite change in $f(x_A) = x_{an}$, so $x = a$ is stable if $\frac{df}{dx} |_{x=a} < 0$; otherwise, $x = a$ is unstable (a repeller) if $\frac{df}{dx} |_{x=a} > 0$. For our function,

$$f(x) = \frac{5}{8} x (4-x) = -\frac{5}{8} x^2 + \frac{5}{2} x,$$

$$\frac{df}{dx} |_{x=a} = -\frac{5}{4} . 0 + \frac{5}{2} = \frac{5}{2} > 0.$$ We find that

$$\frac{df}{dx} |_{x=0} = -\frac{5}{4} \cdot 0 + \frac{5}{2} = \frac{5}{2} > 0 \quad \text{and} \quad \frac{df}{dx} |_{x=\frac{12}{8}} = -\frac{5}{8} \cdot \frac{12}{8} + \frac{5}{2} = -3 + \frac{5}{2} = -\frac{1}{2} < 0.$$ So $x = 0$ is unstable, while $x = \frac{12}{8}$ is stable.

c) If we start with $x_0 = 0.5$, we have

$$x_1 = f(x_0) = \frac{5}{8} \cdot x_0 \cdot (4-x_0) = \frac{5}{8} \cdot 0 \cdot (4-0) = \frac{5}{8} \cdot 0 \cdot \frac{1}{2} = \frac{5}{8} \cdot \frac{1}{2} = \frac{5}{16} = 0.3125.$$

and

$$x_2 = f(x_1) = \frac{5}{8} \cdot x_1 \cdot (4-x_1) = \frac{5}{8} \cdot \frac{5}{16} \cdot (4-\frac{5}{16}) = \frac{5}{8} \cdot \frac{5}{16} \cdot \frac{63}{32} = \frac{5}{8} \cdot \frac{5}{16} \cdot \frac{63}{32} = \frac{16.25}{8} \approx 1.9765.$$

d) The "cobweb diagram" is shown in the answer key.
(a) The Mean Value Theorem states that for a function which is continuous and differentiable on an interval, there is a value of $x$ at which the instantaneous rate of change (the derivative of the function) equals the average rate of change of the function over the interval. If we treat our position during the trip to be a continuous and differentiable function $f(t)$ during the time interval $0 \leq t \leq 4$ hours, the Mean Value Theorem tells us that there is at least one moment (a value of $t = c$ in the interval) at which our speed $v = \left. \frac{df}{dt} \right|_{t=c}$ is the same as the average speed of $\frac{240 \text{ miles}}{4 \text{ hours}} = 60 \text{ mi/hr}$ over the entire interval.

(b) Since the average speed for the entire trip is 60 miles per hour, we may assume that our velocity is a continuous function of time ranging from zero to at least the average speed of 60 miles per hour. The Intermediate Value Theorem states that for a continuous function on an interval, there is a value of $x$ for which the function takes on any value between the minimum value and the maximum value on the interval. So there is at least one moment in the interval $0 \leq t \leq 4$ hours at which our speed was 10 miles per hour (or any other speed between zero and the highest speed).

(c) Although we could make a “common-sense” argument here (as indeed we could for the other two parts of this problem), we need to give a mathematical one. Let us call the position function $f(t)$ and the velocity function $g(t) = \frac{df}{dt}$; we have already said these functions are assumed to be continuous. The Fundamental Theorem of Calculus tells us that $\int_{0}^{t} g(t) \, dt = f(t)$ and that $\int_{a}^{b} g(t) \, dt = f(b) - f(a)$. We know that the total trip distance is $\int_{0}^{4 \text{ hours}} g(t) \, dt = f(4) - f(0) = f(4) - 0 = 240$ miles, where we take the initial position to be $f(0) = 0$. It is likely that there is at least one interval, say, $0 \leq t \leq T$, during which the speed $g(t)$ was less than 60 miles per hour. (There would probably be more than one, but this simplest situation will illustrate the principle of the argument.) We can then write $\int_{0}^{T} g(t) \, dt + \int_{T}^{4} g(t) \, dt = 240$. Since it will be the case that $\int_{0}^{T} g(t) \, dt < 60 \cdot T$, then $\int_{T}^{4} g(t) \, dt > 240 - 60T$.

During the interval $T \leq t \leq 4$ hours, the average speed is $\frac{\int_{T}^{4} g(t) \, dt}{4 - T} > \frac{240 - 60T}{4 - T} = 60 \text{ mi/hr}$, so there must be a time in this interval when the instantaneous speed is greater than 60 miles per hour.
a) The linear approximation to a function \( f(x) \) near a point \( x = a \) is given by

\[
f(x) \approx f(a) + f'(a) \cdot (x - a)
\]

For our function \( f(x) = e^{\sin x} \) in the neighborhood of \( x = \pi \), we need \( f(\pi) = e^{\sin \pi} = e^0 = 1 \) and \( f'(\pi) = \frac{d}{dx} e^{\sin x} \bigg|_{x=\pi} = e^{\sin x} \cdot \cos x \cdot e^{\sin x} \bigg|_{x=\pi} = (-1) \cdot e^0 = -1 \). The linear approximation will then be \( e^{\sin x} \approx 1 + (-1) \cdot (x - \pi) \) near \( x = \pi \).

b) The approximate value of \( e^{\sin 3} \) is then \( f(3) \approx 1 - (3 - \pi) = \pi - 2 \).

This value is \( \pi - 2 \approx 1.14142 \); by calculator, we find that \( e^{\sin 3} \approx 1.15156 \), so the linear approximation is fairly good at \( x = 3 \).

---

The volume of a right circular cylinder is \( V = \pi r^2 h \). If we differentiate this function implicitly with respect to time, we find

\[
\frac{dV}{dt} = \frac{d}{dt} (\pi r^2 h) = \pi \left[ r^2 \frac{dh}{dt} + 2rh \frac{dr}{dt} \right]
\]

We are told that at a particular moment, when \( r = 2 \, \text{m} \) and \( h = 3 \, \text{m} \), that \( \frac{dr}{dt} = 0.5 \, \text{m/min} \) while \( \frac{dV}{dt} = 3 \, \text{m}^3/\text{min} \).

If we solve the rate equation above for \( \frac{dh}{dt} \), we obtain

\[
\frac{1}{\pi} \frac{dV}{dt} = r^2 \frac{dh}{dt} + 2rh \frac{dr}{dt} \Rightarrow \frac{dh}{dt} = \frac{1}{\pi} \frac{dV}{dt} - 2rh \frac{dr}{dt}
\]

\[
\Rightarrow \frac{dh}{dt} = \frac{1}{\pi r^2} \frac{dV}{dt} - \frac{2h}{r} \frac{dr}{dt} = \frac{1}{\pi (2\, \text{m})^2} (3 \, \text{m}^3/\text{min}) - \frac{2(3\, \text{m})}{2\, \text{m}} (0.5 \, \text{m/min})
\]

\[
= \frac{3}{4\pi} - \frac{3}{2} \, \text{m/min} < 0
\]

(since the denominator of the second term is smaller than that of the first)

The height of the cylinder will be decreasing at that time. (It is not necessary to know the exact value of \( \frac{dh}{dt} \).)
The perimeter of this rectangle in the first quadrant is \( p = 2x + 2y \). If one corner of the rectangle lies on the ellipse at \((x, y)\), then
\[
\frac{x^2}{4} + y^2 = 1 \quad \Rightarrow \quad y = \sqrt{1 - \frac{x^2}{4}} \quad \text{(positive square root only)}
\]

The perimeter function can then be written as \( p(x) = 2x + 2\sqrt{1 - \frac{x^2}{4}} \), \( 0 \leq x \leq 2 \).

We can solve for the critical point of this function by setting \( \frac{dp}{dx} = 0 \):
\[
\frac{dp}{dx} = 2 + \frac{1}{2} \cdot \frac{1}{\sqrt{1 - \frac{x^2}{4}}} \cdot \left(-\frac{2x}{4}\right) = 2 - \frac{x}{2\sqrt{1 - \frac{x^2}{4}}} = 0
\]
\[
\Rightarrow \quad x = 4\sqrt{1 - \frac{x^2}{4}} \quad \Rightarrow \quad x^2 = 16\left(1 - \frac{x^2}{4}\right) = 16 - 4x^2
\]
\[
\Rightarrow \quad 5x^2 = 16 \quad \Rightarrow \quad x^2 = \frac{16}{5} \quad \Rightarrow \quad x = \frac{4}{\sqrt{5}} = \frac{4\sqrt{5}}{5}
\]
\[
\Rightarrow \quad y = \sqrt{1 - \frac{(16/5)^2}{4}} = \sqrt{1 - \frac{4}{5}} = \sqrt{\frac{1}{5}} = \frac{\sqrt{5}}{5}
\]

a) If we look at the second derivative of the perimeter function, we find
\[
\frac{d^2p}{dx^2} = \frac{d}{dx} \left( 2 - \frac{1}{2} x \cdot \frac{1}{\sqrt{1 - \frac{x^2}{4}}} \right) = -\frac{1}{2} \left(1 - \frac{x^2}{4}\right)^{-\frac{3}{2}} \cdot \left(-\frac{1}{2}\right) \cdot \left(-\frac{2x}{4}\right) = -\frac{1}{2} \left(1 - \frac{x^2}{4}\right)^{-\frac{3}{2}} - \frac{x^2}{8} \left(1 - \frac{x^2}{4}\right)^{-\frac{3}{2}} < 0,
\]

since \( x^2 \) and \( \frac{1}{\sqrt{1 - \frac{x^2}{4}}} \) are positive. Thus, our critical point is a local maximum for the perimeter function, \( P_{\text{max}} = p\left(\frac{4\sqrt{5}}{5}\right) = 2\left(\frac{4\sqrt{5}}{5}\right) + 2\left(\frac{\sqrt{5}}{5}\right) = \frac{10\sqrt{5}}{5} = 2\sqrt{5} \).

b) We must also check the values of the perimeter function at the endpoints of the interval:
\[
p(0) = 2 \cdot 0 + 2 \cdot \sqrt{1 - \frac{0^2}{4}} = 0 + 2 = 2 \quad \text{and} \quad p(2) = 2 \cdot 2 + 2 \cdot \sqrt{1 - \frac{2^2}{4}} = 4 + 0 = 4.
\]

Since \( 2\sqrt{5} > 4 \), the local maximum of the perimeter function at \( x = \frac{4\sqrt{5}}{5} \) is also the absolute maximum. We now also find that the absolute minimum of \( p(x) \) occurs at \( x = 0 \) and is \( p(0) = 2 \).
a) The Fundamental Theorem of Calculus tells us that for a function \( f(x) \) continuous and differentiable on its domain,
\[
\int_a^x f(t) \, dt = F(x), \text{ where } \frac{df}{dx} = f(x).
\]
So for \( F(x) = \int_{-2}^x \sqrt{\frac{2}{e^u - u}} \, du \),
\[
\frac{dF}{dx} = \sqrt{\frac{2}{e^x - x}}.
\]

b) We also know from the Fundamental Theorem that
\[
\int_a^b f(x) \, dx = F(b) - F(a);
\]
if the upper limit is instead a function \( u(x) \), this becomes
\[
\int_a^{u(x)} f(x) \, dx = F(u(x)) - F(a).
\]
If we now take the derivative with respect to \( x \), we have
\[
\frac{d}{dx} \int_a^{u(x)} f(x) \, dx = \frac{d}{dx} \left[ F(u(x)) - F(a) \right] = \frac{d}{dx} \left[ F(u(x)) \right] - \frac{d}{dx} F(a)
\]
\[
= \frac{d}{du} F(u) \cdot \frac{du}{dx} = F'(u(x)) \cdot \frac{du}{dx} \quad \text{[known as Leibniz' Rule]}
\]

For \( F(x) = \int_x^{\sin 2x} e^t \, dt \), we must re-write this as two separate integrals
before working out the derivative:
\[
F(x) = \int_x^a e^t \, dt + \int_a^{\sin 2x} e^t \, dt
\]
\[
= -\int_x^a e^t \, dt + \int_a^{\sin 2x} e^t \, dt
\]
\[
\Rightarrow \frac{dF}{dx} = - (e^{x^2}) + e^{(\sin 2x)^2} \cdot \frac{d}{dx} (\sin 2x)
\]
apply result from part (a)\]
apply result from part (b)
\[
= 2 \cos 2x \cdot e^{\sin^2 2x} - e^{x^2}.
\]

G. Ruffa -- April 2006