1. This limit can be evaluated by using the "Limit Laws":
\[
\lim_{x \to \infty} \frac{e}{1 + \sqrt[2]{e^x}} = \frac{e}{1 + \sqrt[2]{\lim_{x \to \infty} e^x}} = \frac{e}{1 + \sqrt[2]{0}} = e.
\]

2. Since the only related "limit law" we know is \(\lim_{u \to 0} \frac{\sin u}{u} = 1\), where \(u\) may be a function of \(x\), we need to rewrite our limit to include a term of that form:
\[
\lim_{x \to 0} \frac{\sin (\pi x)}{\pi x} \cdot \frac{\pi}{\pi} = \lim_{x \to 0} \frac{\pi \cdot \sin (\pi x)}{4 \cdot \pi x} = \lim_{x \to 0} \frac{\sin (\pi x)}{\pi x} = \lim_{x \to 0} \frac{\sin (\pi x)}{\pi x} = \frac{\pi}{4}.
\]

3. The two branches of the definition use functions which are continuous on their domains, so the only place where a discontinuity can occur is where the branches join, at \(x = 1\). A function is continuous at a point \(x = a\) if \(f(a)\) is defined, if the two-sided limit \(\lim_{x \to a} f(x)\) exists, and if \(\lim_{x \to a} f(x) = f(a)\).

It is certainly the case that \(f(1)\) exists: it is defined by the branch of \(f(x)\) for \(x \leq 1\);

Thus \(f(1) = a-2 \cdot 1 = a-2\), which is also the limit "from below", \(\lim_{x \to 1^-} f(x) = a-2\).

The limit "from above" at \(x = 1\) will be defined by the branch for \(x > 1\), so
\[
\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} \frac{1}{x} = \frac{1}{1} = 1.
\]

For the two-sided limit \(\lim_{x \to 1} f(x)\) to exist, then, we must have \(\lim_{x \to 1} f(x) = a-2 = \lim_{x \to 1^+} f(x) = 1\), so \(a = 3\). The requirement for continuity \(\lim_{x \to 1} f(x) = f(1)\) will also be satisfied.

4. If the function \(f(x)\) is differentiable and has an inverse \(g(x) = f^{-1}(x)\), the derivative of \(g(x)\) can be found by using the property of an inverse, that \(g(f(x)) = x\). By differentiating implicitly with respect to \(x\), we find
\[
\frac{d}{dx} g(f(x)) = \frac{d}{dx} x \Rightarrow \frac{dg}{dx} \cdot \frac{du}{dx} = 1 \Rightarrow \frac{dg}{dx} \bigg|_{u(x)} \cdot \frac{d}{dx} f(x) = 1
\]

\[
\Rightarrow g'(f(x)) \cdot f'(x) = 1 \Rightarrow g'(f(x)) = \frac{1}{f'(x)}
\]

For Problem 4, we can find \(g'(2)\) because we know that \(f(2) = 2\).

Thus \(g'(2) = g'(f(3)), \) so we know \(x = 3\) because \(f(3) = \frac{1}{2}\), we have \(g'(2) = \frac{1}{f'(3)} = \frac{1}{2} = 5\).\]

5. On the other hand, in order to find \(g'(3)\), we need to know what value of \(x\) leads to \(f(x) = 3\), information which the statement of Problem 5 does not provide.
A straight line on a graph would have the equation \[ y = mx + b \]. If we take the logarithm of both sides of the radioactive decay equation, we find

\[ W(t) = W_0 e^{-\lambda t} \Rightarrow \ln W(t) = \ln (W_0 e^{-\lambda t}) = \ln W_0 + \ln (e^{-\lambda t}) = \ln W_0 + (-\lambda t) \].

Since \( \ln W_0 \) and \( \lambda \) are constants, this equation has a suitable form for a line if we identify
\[ \frac{\ln W(t)}{\ln W_0} = \frac{-\lambda t}{\lambda} + \frac{\ln W_0}{\ln W_0} \]

the graph would then have a logarithmic scale on the vertical axis for \( W(t) \) and a linear scale on the horizontal axis for time \( t \) (a so-called "log-linear" or "semi-log" plot).

Since we have a graph of the derivative \( f'(x) \), we can read the values for which \( f'(x) = 0 \). This tells us that \( f(x) \) has critical points at \( x = 1 \) and \( x = 3 \). If we look at the slope of the curve for \( f'(x) \), we can at least tell what sign that slope has, which tells us the sign of the rate of change of \( f(x) \), which is the sign of the second derivative \( f''(x) \).

We see that the slope of \( f'(x) \) at \( x = 2 \) is zero, while the value of \( f'(x) \) there is not zero. So the graph for the function \( f(x) \) has an inflection point at \( x = 2 \).

The slope of \( f(x) \) at \( x = 3 \) is negative, so \( f''(x) < 0 \) at \( x = 3 \); so this critical point is a local maximum.

If we examine this graph of the function \( f(x) \), we see that its slope is negative (or that its value decreases) across the entire interval; thus \( f'(x) < 0 \) on the interval depicted. The slope of \( f(x) \), however, becomes steadily less negative as \( x \) increases, so the value of \( f'(x) \) is increasing or its rate of change, \( f''(x) \) is positive. (Alternatively, we may notice that \( f(x) \) is concave upward on this interval.) Hence, \( f''(x) > 0 \) on this graph.

Since \( N(t) \) indicates the size of the population as a function of time, the growth rate \( g(t) \) is then the time derivative \( \frac{dN}{dt} \). The "per capita growth rate" is the growth rate per member of the population or \( \frac{dN}{dt} / N \). For our growth equation, this will be

\[ \frac{1}{N} \frac{dN}{dt} = \frac{1}{N} g(N) = \frac{N(2N-1)(4-N)}{N} = (2N-1)(4-N) \].
The function relating leaf stem diameter $y$ to leaf area $x$ is $y = f(x) = \frac{1}{10} \cdot x^{-\frac{5}{9}}$, so the rate of change is $\frac{dy}{dx} = \frac{1}{10} \cdot \frac{5}{9} \cdot x^{-\frac{16}{9}}$. We are interested here in fractional errors of measurement, however. A 10% uncertainty (or "error") in the measurement of $x$ is a fractional error of $\frac{dx}{x} = 0.1$; we wish to find the corresponding fractional error for the computed value of $y$, $\frac{dy}{y}$. From our equation for the derivative, we find

$$\frac{dy}{y} = \frac{1}{10} \cdot \frac{5}{9} \cdot x^{-\frac{16}{9}} \cdot \frac{dx}{x} = \frac{1}{10} \cdot \frac{5}{9} \cdot x^{-\frac{16}{9}} \cdot (0.1 x) = \frac{5}{900} \cdot x^{-\frac{16}{9}},$$

so the fractional error in $y$ is

$$\frac{dy}{y} = \frac{5}{900} \cdot x^{-\frac{16}{9}} = \frac{5}{90} \cdot x^{-\frac{16}{9}},$$

from which we find the percentage error in $y$ to be $(\frac{dy}{y}) \cdot 100\% = \frac{5}{90} \cdot 100\% = \frac{5}{9} \%$. \(E\)

The closest choice listed is \(B\): 5.4 \%, so unless that is a typographical error, none of the given choices is correct.

We know that $\int_a^b f(x) \, dx + \int_b^c f(x) \, dx = \int_a^c f(x) \, dx$, so it will be the case that $\int_a^2 f(x) \, dx + \int_2^3 f(x) \, dx = \int_1^3 f(x) \, dx$. If we substitute the values given for the definite integrals in the problem, we have $\int_a^2 f(x) \, dx + 1 = -1 \Rightarrow \int_1^2 f(x) \, dx = -2$. \(A\)

If we simply follow the "Limit Laws", we obtain $\lim_{x \to 0} \frac{\sin \pi x}{x^2 + 2x} = \frac{\sin (\pi \cdot 0)}{2 + 2 \cdot 0} = \frac{\pi}{2 \cdot 2} = \frac{\pi}{4}$, which is an indeterminate ratio. It is in a suitable form, however, for the application of L'Hôpital's Rule:

$$\lim_{x \to 0} \frac{\sin \pi x}{x^2 + 2x} = \lim_{x \to 0} \frac{(\sin \pi x)'}{(x^2 + 2x)'} = \frac{\pi \cdot \cos (\pi \cdot 0)}{2 \cdot 2} = \frac{\pi}{4}.$$

b) The limit $\lim_{x \to \infty} x^2 e^{-2x}$ produces the indeterminate product $\infty \cdot 0$. We will need to re-write this as an indeterminate ratio:

$$\lim_{x \to \infty} x^2 e^{-2x} = \lim_{x \to \infty} \frac{x^2}{e^{2x}} = \lim_{x \to \infty} \frac{(x^2)'}{(e^{2x})'} = \lim_{x \to \infty} \frac{2x}{4e^{2x}} = \frac{\infty}{\infty}.$$

Since we still have an indeterminate ratio, we shall apply L'Hôpital's Rule again:

$$\lim_{x \to \infty} \frac{2x}{4e^{2x}} = \lim_{x \to \infty} \frac{(2x)'}{(4e^{2x})'} = \lim_{x \to \infty} \frac{2}{4e^{2x}} = \frac{2}{\infty} = 0.$$
a) It will help a bit if we first write all the terms in the function using exponents:

\[ f(x) = \sqrt{x} - \frac{1}{x^3} - x^4 \]
\[ = x^{\frac{1}{2}} - x^{-3} - x^4 \]
\[ \Rightarrow f'(x) = \frac{1}{2}x^{-\frac{1}{2}} - (-3)x^{-4} - 4x^3 \]
\[ = \frac{1}{2}x^{-\frac{1}{2}} + 3x^{-4} - 4x^3 \]

or \[ \frac{1}{2\sqrt{x}} + \frac{3}{x^4} - 4x^3 \]

b) This differentiation will require the Product Rule and the Chain Rule:

\[ f(x) = x^2e^{-2x} \Rightarrow f'(x) = \frac{d}{dx}(x^2) \cdot e^{-2x} + x^2 \cdot \frac{d}{dx}(e^{-2x}) \]
\[ = 2x \cdot e^{-2x} + x^2 \cdot (-2x) \cdot e^{-2x} \]
\[ = 2xe^{-2x} - 2xe^{-2x} = (2x - 2x^2) e^{-2x} \]

b) The Quotient Rule is of use here:

\[ f(x) = \frac{x^2}{5-2x^2} \Rightarrow f'(x) = \frac{(5-2x^2) \cdot \frac{d}{dx}(x^2) - (x^2) \cdot \frac{d}{dx}(5-2x^2)}{(5-2x^2)^2} \]
\[ = \frac{(5-2x^2)(2x) - (x^2)(-4x)}{(5-2x^2)^2} \]
\[ = \frac{10x - 4x^3 + 4x^3}{(5-2x^2)^2} = \frac{10x}{(5-2x^2)^2} \]

b) To calculate this derivative, we will need to recall or derive the derivative function of an exponential function with a base other than \(e\):

If we write \( y = 3^x \), then we can use logarithmic differentiation to find

\[ \ln y = x \cdot \ln 3 \Rightarrow \frac{d}{dx} (\ln y) = \frac{d}{dx} [(\ln 3) \cdot x] \]
\[ \Rightarrow \frac{1}{y} \cdot \frac{dy}{dx} = (\ln 3) \cdot \frac{d}{dx}(x) = (\ln 3) \cdot 1 \]
\[ \Rightarrow \frac{dy}{dx} = (\ln 3) \cdot y = (\ln 3) \cdot 3^x \]

We can now differentiate our function:

\[ f(x) = x^3 - 3^{-x} \Rightarrow f'(x) = -3x^{-4} - \frac{d}{dx} (3^{-x}) \]
\[ = -3x^{-4} - \frac{d}{dx}(3^x) \cdot -x \]
\[ = -3x^{-4} - (\ln 3) \cdot 3^x \cdot (-1) \]
\[ = -3x^{-4} + (\ln 3) \cdot 3^{-x} \]
The Chain Rule is needed for these:\n\[ f(x) = \exp\left[ -\sin (1+x^3) \right] = e^{-\sin (1+x^3)} \]
\[ \Rightarrow f'(x) = \frac{d}{du} e^u \cdot \frac{d}{dx} u = e^u \cdot \frac{d}{dx} \left[ -\sin (1+x^3) \right] \]
\[ = e^u \cdot \frac{d}{dv} (-\sin v) \cdot \frac{d}{dx} v \]
\[ = e^u \cdot (-\cos v) \cdot \frac{d}{dx} (1+x^3) \]
\[ = e^u \cdot (-\cos v) \cdot 3x^2 = -3x^2 \cdot \cos (1+x^3) \cdot e^{-\sin (1+x^3)} \]

b) \[ f(x) = \ln (\sec x) \]
\[ \Rightarrow f'(x) = \frac{d}{du} \ln u \cdot \frac{d}{dx} u = \frac{1}{u} \cdot \frac{d}{dx} (\sec^2 x) = \frac{1}{u} \cdot \frac{d}{dv} (v^2) \cdot \frac{d}{dx} v \]
\[ = \frac{1}{u} \cdot 2v \cdot \frac{d}{dx} (\sec x) \]
\[ = \frac{1}{u} \cdot 2v \cdot \sec x \tan x \]
\[ = \frac{1}{\sec^2 x} \cdot 2 \sec x \cdot \sec x \tan x \]
\[ = 2 \tan x \]

This is also a Chain Rule problem:\n\[ f(x) = g(e^{2x}) \Rightarrow f'(x) = \frac{d}{dx} g(u) = \frac{d}{du} g(u) \cdot \frac{d}{dx} u = \frac{d}{du} g(u) \bigg|_{u=e^{2x}} \cdot \frac{d}{dx} (e^{2x}) \]
\[ = g'(u) \cdot 2e^{2x} \]

Since this equation involves y in a way where it cannot be algebraically separated, we must find the derivative \( \frac{dy}{dx} \) through implicit differentiation:\n\[ y = \ln (x^2+y) \Rightarrow \frac{d}{dx} (y) = \frac{d}{dx} \left[ \ln (x^2+y) \right] \]
\[ \Rightarrow \frac{dy}{dx} = \frac{1}{u} \cdot \frac{d}{dx} (x^2+y) \]
\[ = \frac{1}{x^2+y} \cdot (2x + \frac{dy}{dx}) \]

We can rearrange this algebraically to give:
\[ \frac{dy}{dx} - \frac{1}{x^2+y} \cdot \frac{dy}{dx} = \frac{2x}{x^2+y} \Rightarrow (\frac{x^2+y-1}{x^2+y}) \cdot \frac{dy}{dx} = \frac{2x}{x^2+y} \]
\[ \Rightarrow \frac{dy}{dx} = \frac{2x}{x^2+y-1} \]
a) The slope of the curve \( y = f(x) \) at \( x = a \) is given by \( f'(a) \). The tangent line to the curve at that point is taken as a linear approximation to how the curve behaves in the neighborhood of \( x = a \). Since the tangent line passes through the point \((a, f(a))\), its equation is \((y - f(a)) = f'(a) \cdot (x - a)\) [point-slope form of equation]

\[ y = f(a) + f'(a) \cdot (x - a). \]

As the linear approximation to the curve, we then write

\[ f(x) \approx f(a) + f'(a) \cdot (x - a). \]

For our function, \( f(x) = (1-x)^r, r > 0 \), we have \( f'(x) = r \cdot (1-x)^{r-1} \frac{d}{dx}[1-x] = -r \cdot (1-x)^{r-1} \). In the vicinity of \( x = a = 0 \), then, our linear approximation to \( f(x) \) is

\[ f(x) = (1-x)^r \approx (1-0)^r + [-r \cdot (1-0)^{r-1}] \cdot (x - 0) = 1 - rx \]

\[ f(a) + f'(a) \cdot (x - a) \]

b) To use this approximation to estimate 0.998\(^{100}\), we must put this into correspondence with our function: \((0.998)^{100}\). Thus \( r = 100 \) and \( 1-x = 0.998 \Rightarrow x = 0.002 \).

\[ (1-x)^r \]

Our estimate from the linear approximation is then \( 0.998^{100} = 1 - rx = 1 - 100 \cdot 0.002 = 1 - 0.2 = 0.8 \).

(from calculator: 0.81857...)

20 For a rational function, the vertical asymptotes are located where the denominator becomes zero, provided the numerator is not also zero there. For our function, \( f(x) = \frac{x-2}{2x-1} \), the denominator is zero for \( 2x-1 = 0 \Rightarrow x = \frac{1}{2} \). The numerator is not zero there, so \( x = \frac{1}{2} \) is the single vertical asymptote. The horizontal asymptote is the function's "limit at infinity". Here, we have

\[ \lim_{x \to \infty} \frac{x-2}{2x-1} = \frac{1}{2} \] (by the "division method" or L'Hôpital's Rule), so \( y = \frac{1}{2} \) is the horizontal asymptote.
a) Since the first derivative $f'(x)$ gives the rate of change of the function $f(x)$, local extrema (maxima or minima) occur where $f'(x) = 0$ and

$$f(x) \begin{cases} \text{increasing} & \text{where } f'(x) > 0 \\ \text{decreasing} & \text{where } f'(x) < 0 \end{cases}.$$  

For our function, $f'(x) = 6x^2 - 6 = 6(x^2 - 1) = 0 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1$;

$6(x^2 - 1) > 0 \Rightarrow x^2 > 1 \Rightarrow x < -1 \text{ or } x > 1$; $6(x^2 - 1) < 0 \Rightarrow x^2 < 1$  

$-1 < x < 1$.

Thus, $f(x)$ is increasing on the intervals $(-\infty, -1)$ and $(1, \infty)$ and is decreasing on the interval $(-1, 1)$. This tells us that $x = -1$ is a local maximum and $x = 1$ is a local minimum.

b) The second derivative $f''(x)$ gives the rate of change of $f'(x)$, so $f''(x) > 0$ shows an increasing slope and $f''(x) < 0$, a decreasing slope, or

$f(x)$ is concave $\begin{cases} \text{upward} & \text{where } f''(x) > 0 \\ \text{downward} & \text{where } f''(x) < 0 \end{cases}$.

An inflection point occurs where $f''(x) = 0$ provided $f''(x)$ is not also zero there (if $f'(x)$ is also at such a point, that location is not necessarily an inflection point).

For our function, $f''(x) = 12x$, so $f(x)$ is concave upward for $x > 0$

and concave downward for $x < 0$. At $x = 0$, $f'(x) = 0$ but $f''(x) = 0$, so

this is an inflection point. We may also note that $f(x)$ is concave downward

at $x = -1$ and concave upward at $x = 1$, confirming their identifications as

local maximum and minimum, respectively.

22 The total amount of fence required is that

for four segments of length $W$ and just one of length $L$ (since the enclosures are built against a wall). So the amount of fencing material available sets a restriction of

$p = L + 4W = 200$ ft.

We wish to maximize the enclosed area

$A = LW$.

We can make this a function of just one variable by eliminating, say, $L$, by rewriting the parameter equation as

$L = 200 - 4W$.

The area equation then becomes $A = (200 - 4W)W = 200W - 4W^2$. We can find the critical point of this function by setting $\frac{dA}{dW} = 0$ and solving for $W$:

$\frac{dA}{dW} = 200 - 8W = 0 \Rightarrow W = \frac{200}{8} = 25$ ft. \Rightarrow L = 200 - 4(25) = 100$ ft.

The maximum area, which can be enclosed is then $A = (25 \text{ ft})(100 \text{ ft}) = 2500$ ft$^2$.

If we look at the second derivative of the area function, we find $\frac{d^2A}{dW^2} = -8 < 0$.

So the area function is always concave downward: our critical point is a local maximum.
In this problem, we are only being asked to show that the function \( W(t) = W_0 e^{-\lambda t} \) is a solution for the given differential equation, not to solve the differential equation itself.

Upon substitution, we find that
\[
\frac{d}{dt} W(t) = \frac{d}{dt} (W_0 e^{-\lambda t}) = W_0 \cdot \frac{d}{dt} (e^{-\lambda t}) = W_0 \cdot \frac{d}{dt} e^{u} \cdot \frac{d}{dt} u
\]
\[
= W_0 \cdot e^{u} \cdot \frac{d}{dt} (-\lambda t) = W_0 \cdot e^{u} \cdot (-\lambda)
\]
\[
= -\lambda \cdot W_0 e^{-\lambda t} = -\lambda W(t)
\]

Here, we actually are to solve the differential equation. It is separable, so we can write
\[
\frac{dy}{dx} = e^{-0.3x} \Rightarrow \int dy = \int e^{-0.3x} \, dx
\]
\[
\Rightarrow y = \int e^{u} \cdot (-\frac{1}{0.3} \, du) \Rightarrow du = -0.3 \, dx
\]
\[
\Rightarrow dx = \frac{-1}{0.3} \, du
\]
\[
y = -\frac{1}{0.3} \cdot e^{-0.3x} + C
\]

We can also find the value of the integration constant, subject to the "initial-value" condition
\[
y(0) = 6 : \quad y = -\frac{1}{0.3} \cdot e^{-0.3 \cdot 0} + C = 6
\]
\[
\Rightarrow C = 6 + \frac{1}{0.3} \cdot 1 = 6 + \frac{1}{0.3} = 6 + \frac{10}{3} = \frac{28}{3}
\]

The solution function is thus \( y = \frac{28}{3} - \frac{10}{3} e^{-0.3x} \).

We can rearrange the differential equation \( \frac{dN}{dt} = 2N (1 - \frac{N}{b}) \) in terms of differentials:
\[
dN = 2N (1 - \frac{N}{b}) \, dt
\]

If we look at small changes, rather than infinitesimal ones, this becomes a linear approximation
\[
\Delta N \approx 2N (1 - \frac{N}{b}) \Delta t \Rightarrow N(t) - N(0) \approx 2 \cdot N(0) \cdot (1 - \frac{N(0)}{b}) \cdot (t-0)
\]

We are told that \( N(0) = 2 \), so we can estimate the population at \( t = 0.1 \):
\[
N(0.1) \approx N(0) + 2 \cdot N(0) \cdot (1 - \frac{N(0)}{b}) \cdot (0.1-0)
\]
\[
= 2 + 2 \cdot 2 \cdot (1 - \frac{2}{6}) \cdot 0.1 = 2 + 2 \cdot 2 \cdot \frac{2}{3} \cdot \frac{1}{10} = \frac{68}{30} = \frac{34}{15}
\]
\[
\approx 2.267
\]
(26) This integral can be determined through a "u-substitution":

\[
\int \sin \left(3\pi x\right) \, dx = \frac{u}{3\pi} \\
\Rightarrow \int \sin u \cdot \left(\frac{1}{3\pi} \, du\right) = \frac{1}{3\pi} \int (-\cos u) \, du \\
= \frac{1}{3\pi} \cos (3\pi x) + C
\]

(27) The important feature we are asked to recognize for this problem is that the radical term \( y = \sqrt{4-x^2} \)
describes the first quadrant portion of the figure.

\[
y = \sqrt{4-x^2} \Rightarrow y^2 = 4-x^2 \Rightarrow x^2 + y^2 = 4, \\
a circle of radius 2.
\]

The integral then describes the negative of the area between a square of side length 2 and a quarter-circle of radius 2, or

\[
\int_0^2 \sqrt{4-x^2} - 2 \, dx = \frac{1}{4} \left(\pi (2)^2\right) - 2^2 = \pi - 4 < 0.
\]

(28) This definite integral can be computed using a "u-substitution":

\[
\int_0^5 e^{\frac{u}{3}} \, dx = \frac{u}{9x} \\
\Rightarrow \int_0^5 e^u \cdot \left(\frac{1}{3} \, du\right) = \frac{1}{3} \int_0^5 e^u \, du = \frac{1}{3} \left[\left.e^u\right|_0^5\right] = \frac{1}{3} (e^5 - e^0) = \frac{1}{3} (e^5 - 1)
\]
The Fundamental Theorem of Calculus tells us that for a function \( f(x) \) continuous on an interval, \( a \leq x \leq b \),
\[
\int _a^b f(x) \, dx = F(b) - F(a), \quad \text{where} \quad \frac{d}{dx} F(x) = f(x).
\]
We can extend this result to the case of having a function \( u(x) \) as the upper limit:
\[
\int _a^{u(x)} f(x) \, dx = F(u(x)) - F(a).
\]
The derivative of this integral will then be
\[
\frac{d}{dx} \int _a^{u(x)} f(x) \, dx = \frac{d}{dx} F(u(x)) - \frac{d}{dx} F(a)
\]
\[
= \frac{d}{du} F(u) \cdot \frac{du}{dx} - 0 \quad \text{Chain Rule}
\]
\[
= \left. \frac{d}{dx} F(x) \right|_{u(x)} \cdot \frac{du}{dx} = f(u(x)) \cdot \frac{du}{dx}. \quad \text{[Leibniz' Rule]}
\]
For our function \( f(x) = \frac{d}{dx} \int _0^{\ln x} e^{2u} \, du \), we will obtain
\[
f(x) = e^{2(\ln x)} \cdot \frac{d}{dx} (\ln x) = e^{2\ln x} \cdot \frac{1}{x}
\]
\[
= (e^{\ln x})^2 \cdot \frac{1}{x} = x^2 \cdot \frac{1}{x} = x
\]
Note: \( u \) here is not the same as the \( u \) used in the derivation above.

This integral can be computed by using a "u-substitution":
\[
\int _0^{\pi/3} \sec^2 \left( \frac{x}{3} \right) \, dx
\]

\[
\begin{array}{c|c|c}
\quad & u = \frac{x}{3} & x = \pi/3 \\
\hline
0 & 0 & \pi/3 \\
\end{array}
\]

\[
\Rightarrow \quad du = \frac{1}{3} \, dx, \quad dx = 3 \, du
\]

\[
= 3 \int _0^{\pi/3} \sec^2 u \, du = 3 \tan u \bigg| _0^{\pi/3} = 3 \tan \left( \frac{\pi}{3} \right) - 3 \tan 0
\]
\[
= 3 \cdot \sqrt{3} - 3 \cdot 0 = 3\sqrt{3}
\]
These integrals can both be solved by using "u-substitutions".

a) \[ \int \frac{3}{x-2} \frac{1}{u} \, du \quad u = x - 2 \quad \Rightarrow \quad du = dx \]

\[ \Rightarrow \quad 3 \int \frac{1}{u} \, du = 3 \ln |x| + C \quad \Rightarrow \quad 3 \ln |x-2| + C \]

b) \[ \int 3x \cdot e^{x^2} \, dx \quad u = x^2 \quad \Rightarrow \quad du = 2x \, dx \]

\[ \Rightarrow \quad x \, dx = \frac{1}{2} \, du \]

\[ \Rightarrow \quad 3 \int e^u \cdot \left( \frac{1}{2} \, du \right) = \frac{3}{2} \int e^u \, du = \frac{3}{2} e^u + C \quad \Rightarrow \quad \frac{3}{2} e^{x^2} + C. \]

For the portion of the curves that we are interested in, the line \( x + y = 3 \Rightarrow y = 3 - x \) is above the hyperbola \( y = \frac{1}{x} \). To find the limits of integration, we need to determine the points of intersection of these two curves. If we set the functions equal to one another, we find that

\[ y = 3 - x = \frac{1}{x} \]

\[ \Rightarrow \quad 3x - x^2 = 1 \quad \Rightarrow \quad x^2 - 3x + 1 = 0, \quad x \neq 0 \]

for which the quadratic formula gives \( x = \frac{3 \pm \sqrt{(3)^2 - 4(1)(1)}}{2} = \frac{3 \pm \sqrt{5}}{2} \). The integral for the area bounded by these two curves is then

\[ A = \int_{\frac{3-\sqrt{5}}{2}}^{\frac{3+\sqrt{5}}{2}} (3 - x) - \frac{1}{x} \, dx. \]

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