1) An equation for the tangent plane can be found once we know the direction of a normal vector $n = <a, b, c>$ for the plane; that equation is then given by $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$, where $(x_0, y_0, z_0)$ is a point in the plane. For a “level surface” given by a function $F(x, y, z) = c$, the normal vector to the surface at the point $(x_0, y_0, z_0)$ is given by $n = \hat{\nabla}F_{(x_0, y_0, z_0)}$.

So, for our surface $xz^2 - 2y^3 + 1 = 0$ (which may be interpreted as either the level surface $xz^2 - 2y^3 + 1 = 0$ for the function $F = xz^2 - 2y^3 + 1$ or the surface $xz^2 - 2y^3 = -1$ for the function $F = xz^2 - 2y^3$), we obtain

$$\hat{\nabla}F = \begin{pmatrix} \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} \end{pmatrix} = \begin{pmatrix} z^2 \frac{\partial}{\partial x} x - 0, 0 - 2 \frac{\partial F}{\partial y} y^3, x \frac{\partial F}{\partial z} z^2 - 0 \end{pmatrix}$$

$$= \begin{pmatrix} z^2 \cdot 1, -6y^2, x \cdot 2z \end{pmatrix} = \begin{pmatrix} z^2, -6y^2, 2xz \end{pmatrix}$$

$$\Rightarrow n = \hat{\nabla}F\bigg|_{(1,1,1)} = \begin{pmatrix} z^2, -6y^2, 2xz \end{pmatrix}\bigg|_{(1,1,1)}$$

$$= \begin{pmatrix} 1^2, -6 \cdot 1^2, 2 \cdot 1 \cdot 1 \end{pmatrix} = \begin{pmatrix} 1, -6, 2 \end{pmatrix}.$$ 

The equation of the tangent plane to this surface at $(1, 1, 1)$ is thus

$$1(x - 1) + (-6)(y - 1) + 2(z - 1) = x - 1 - 6y + 6 + 2z - 2 = 0$$

$$\Rightarrow x - 6y + 2z + 3 = 0 \quad \text{or} \quad x - 6y + 2z = -3 \quad \text{(D)}.$$ 

It might be mentioned that this equation also represents the “linearization” of the function $F(x, y, z)$ in the neighborhood of $(1, 1, 1)$.

2) For a space curve $r(t)$, the unit tangent vector at the point corresponding to $t = t_0$ is given by $T(t_0) = \frac{\frac{d}{dt}r(t)}{\left|\frac{d}{dt}r(t)\right|}_{t=t_0}$. * So for our space curve,

$$r(t) = \begin{pmatrix} e^t + e^{-t} + 1, e^t - e^{-t} + 3, t^2 - 4t + 5 \end{pmatrix},$$

(continued)

* In some other books, and in other fields such as physics, the unit tangent vector is denoted by $\hat{T}$, to make it clear that this is a unit vector.
we find
\[
\mathbf{r}'(t) = \frac{d}{dt} \mathbf{r}(t) = \frac{d}{dt} \left\{ e^t + e^{-t} + 1, e^t - e^{-t} + 3, t^2 - 4t + 5 \right\} \\
= \left\{ e^t - e^{-t}, e^t - (-e^{-t}), 2t - 4 \right\}
\]
\[
\Rightarrow \mathbf{r}'(1) = \left\{ e^1 - e^{-1}, e^1 + e^{-1}, 2 \cdot 1 - 4 \right\} = \left\{ e - e^{-1}, e + e^{-1}, -2 \right\}
\]
\[
\Rightarrow |\mathbf{r}'(1)| = \left[ \left( e - e^{-1} \right)^2 + \left( e + e^{-1} \right)^2 + \left( -2 \right)^2 \right]^{1/2} \\
= \left[ \left( e^2 - 2 + e^{-2} \right) + \left( e^2 + 2 + e^{-2} \right) + 4 \right]^{1/2} \\
= \left( 2e^2 + 4 + 2e^{-2} \right)^{1/2} = \sqrt{2} \cdot (e^2 + 2 + e^{-2})^{1/2} \\
= \sqrt{2} \cdot [(e + e^{-1})^2]^{1/2} = \sqrt{2} \cdot (e + e^{-1})
\]
\[
\Rightarrow \mathbf{T}(1) = \frac{\mathbf{r}'(1)}{|\mathbf{r}'(1)|} = \frac{1}{\sqrt{2} \left( e + e^{-1} \right)} \left\{ e - e^{-1}, e + e^{-1}, -2 \right\} .
\]

3) Differentiation of a function of more than one variable follows the same basic method as for differentiation of functions of a single variable, but the techniques must be extended. For composite functions of which use just one variable, the Chain Rule is
\[
\frac{d}{dt} f(x(t)) = \frac{df}{dx} \cdot \frac{dx}{dt} ,
\]
but for, say, a function of two variables, this becomes
\[
\frac{d}{dt} f(x(t), y(t)) = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt} ;
\]
if \( x \) and \( y \) are themselves functions of more than one variable, we use the appropriate partial derivative of each instead.

In this Problem, we have the function \( u = f(x(t), y(t)) \), so we will be working with this “two-variable Chain Rule”:

\[
\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial s} *
\]

\[
= \frac{\partial}{\partial x} \left( x^3 + y^3 \right) \cdot \frac{\partial}{\partial s} \left( e^{ts} + 1 \right) + \frac{\partial}{\partial y} \left( x^3 + y^3 \right) \cdot \frac{\partial}{\partial s} \left( e^{t+s} + 1 \right) \\
= \left( 3x^2 + 0 \right) \cdot \frac{\partial}{\partial s} \left( e^{ts} \right) + \frac{\partial}{\partial y} \left( 0 + 3y^2 \right) \cdot \frac{\partial}{\partial s} \left( e^t \cdot e^s \right) \\
= 3x^2 \cdot e^{ts} + 3y^2 \cdot e^t \cdot e^s = 3 \left( e^{ts} + 1 \right) \cdot e^{ts} + 3 \left( e^{t+s} + 1 \right) \cdot e^t \cdot e^s .
\]

* The other partial derivative of \( u \) will be \( \frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial t} \) (continued)
Since we wish to evaluate this derivative at specific values of $s$ and $t$, it is not necessary to take time arranging the expression into an algebraically simpler form; hence,

$$
\frac{\partial u}{\partial s} \bigg|_{s=0, t=1} = [3(e^{t^s} + 1)^2 \cdot te^{t^s} + 3(e^{t^s} + 1)^2 \cdot e^t \cdot e^s] \bigg|_{s=0, t=1}
$$

$$= 3(e^{1^0} + 1)^2 \cdot 1 \cdot e^{1^0} + 3(e^{1^0} + 1)^2 \cdot e^1 \cdot e^0
$$

$$= 3(1+1)^2 \cdot 1 \cdot 1 + 3(3^1 + 1)^2 \cdot e \cdot 1 = 12 + 3e(3^1 + 1)^2 .
$$

(D)

4) We locate critical points for functions of more than one variable in much the same way as we do for functions of a single variable: the chief difference is that we look for points at which all of the first partial derivatives equal zero at once. For our function, $f(x, y) = x^2 + 2y^3 - 6xy + 18y - 2x + 11$, we will need to solve simultaneously

$$\frac{\partial f}{\partial x} = 2x + 0 - 6y + 0 - 2 + 0 = 2x - 6y - 2 = 0 \quad \text{and}$$

$$\frac{\partial f}{\partial y} = 0 + 6y^2 - 6x + 18 - 0 + 0 = 6y^2 - 6x + 18 = 0 .$$

The most direct approach would be to solve the first equation for $x$ and insert the result into the second equation, resulting in a quadratic equation in $y$:

$$2x - 6y - 2 = 0 \Rightarrow x = 3y + 1
$$

$$\Rightarrow 6y^2 - 6(3y + 1) + 18 = 0 \Rightarrow 6y^2 - 18y + 12 = 0
$$

$$\Rightarrow 6(y - 1)(y - 2) = 0 \Rightarrow y = 1, 2 \Rightarrow x = 4, 7 .$$

We have found two critical points for our function at $(4, 1)$ and $(7, 2)$ [this, incidentally, eliminates choice (A)], but we still need to classify them. For this, we carry out a procedure analogous to the Second Derivative Test for functions of one variable. At each critical point, we calculate an index value from the second partial derivatives, $D = f_{xx} \cdot f_{yy} - (f_{xy})^2$; the result corresponds to the types according to

$$D > 0 , f_{xx} > 0 -- \text{local minimum} ;$$

$$D > 0 , f_{xx} < 0 -- \text{local maximum} ;$$

$$D < 0 -- \text{“saddle point” (minimum in one dimension, maximum in the other)} .$$

For the function in question here, we find

(continued)
\[
f_{xx} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x} (2x - 6y - 2) = 2 ,
\]
\[
f_{yy} = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y} (6y^2 - 6x + 18) = 12y , \text{ and}
\]
\[
f_{xy} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y} (2x - 6y - 2) = -6 \quad (= f_{yx} = \frac{\partial^2 f}{\partial x \partial y} \text{ also}) ,
\]

leading to the index value \( D = (2)(12y) - (-6)^2 = 24y - 36 \). For the critical point \((4,1)\), we have \( D = 24 \cdot 1 - 36 < 0 \), so this is a saddle point. For \((7,2)\), we find \( D = 24 \cdot 2 - 36 > 0 \), indicating that it is a local extremum; since \( f_{xx} = 2 > 0 \) in any case, this critical point is a local minimum.  

\[ \text{(D)} \]

5) The given integral, \( \int_0^2 \int_{x/2}^{5/x^2+1} f(x,y) \, dy \, dx \), indicates that the function \( f(x,y) \) is to be integrated in the region above the line \( y = \frac{1}{2} x \) and below the curve \( y = \frac{5}{x^2 + 1} \) in the interval from \( x = 0 \) to \( x = 2 \). We can check that the bounding line and curve in fact intersect at \((2,1)\) [it is not so simple to set the two functions equal to one another, \( \frac{1}{2} x = \frac{5}{x^2 + 1} \), in order to then solve \( \frac{1}{2} x^3 + \frac{1}{2} x - 5 = 0 \)]. The two curves also intercept the \( y \)-axis, so \( x = 0 \) also bounds the region of integration. A graph is presented below (while we are not asked in the Problem to plot a graph, a sketch of the situation will be helpful in planning the solution).
If we decide to change the order of integration, we will be integrating with respect to \( y \) first. One thing we will need to do is express the bounding curves as functions of \( y \) instead: \( y = \frac{1}{2} x \Rightarrow x = 2y \), \( y = \frac{5}{x^2 + 1} \Rightarrow x^2 + 1 = \frac{5}{y} \Rightarrow x = \sqrt{\frac{5}{y} - 1} \). The \( y \)-intercept of the line is \((0,0)\) and that of the curve is \((0,5)\), so our interval of integration now runs from \( y = 0 \) to \( y = 5 \). The “lower” bound of the region of integration is \( x = 0 \) throughout, but we must be cognizant of the fact that the bounding line and the bounding curve intersect, so the “upper” bound of the region changes at \((2,1)\). This means that we will have to set up two integrals, one between the \( y \)-axis and the line from \( y = 0 \) to \( y = 1 \), the other between the \( y \)-axis and the curve from \( y = 1 \) to \( y = 5 \), thus

\[
\int_0^1 \int_0^{2y} f(x,y) \, dx \, dy + \int_1^5 \int_0^{\sqrt{\frac{5}{y} - 1}} f(x,y) \, dx \, dy .
\]  

(B)

6) The straightforward way to compute this line integral, \( \int_C \left( 3x^2 + 7y \right) \, dx + \left( 4x + 4y \right) \, dy \), around the closed curve of the “upper semi-circle” of \( x^2 + y^2 = 16 \) is to carry out one calculation along the \( x \)-axis from \( A(\ -4,0) \) to \( B(\ 4,0) \), and a second on the semi-circle, returning to \( A \). Along the path from \( A \) to \( B \), we have \( y = 0 \) and \( dx = 0 \), reducing the calculation to

\[
\int_{A\to B} (3x^2 + 7\cdot0) \, dx + (4x + 4\cdot0) \cdot 0 = \int_{-4}^4 3x^2 \, dx = x^3 \bigg|_{-4}^4 = 64 - (-64) = 128 .
\]

For the semi-circular path back to \( A \), it is easiest to make the integration parametrically by using polar coordinates, which then follows the circle \( r = 4 \) from \( \theta = 0 \) to \( \theta = \pi \). We will therefore make the substitutions \( x = 4 \cos \theta \Rightarrow dx = -4 \sin \theta \, d\theta \) and \( y = 4 \sin \theta \Rightarrow dy = 4 \cos \theta \, d\theta \). Along this path, the line integral becomes

\[
\int_{B\to A} (3x^2 + 7y) \, dx + (4x + 4y) \, dy
\]

\[
= \int_0^\pi (3 \left[ 4 \cos \theta \right]^2 + 7 \cdot 4 \sin \theta) (-4 \sin \theta \, d\theta) + (4 \cdot 4 \cos \theta + 4 \cdot 4 \sin \theta) (4 \cos \theta \, d\theta)
\]

\[
= 16 \int_0^\pi (-12 \cos^2 \theta \sin \theta - 7 \sin^2 \theta + 4 \cos^2 \theta + 4 \sin \theta \cos \theta) \, d\theta .
\]

This integral can now be carried out term-by-term:

\[
\int_0^\pi \cos^2 \theta \sin \theta \, d\theta \rightarrow \int_0^{-1} u^2 (-du) = \frac{1}{3} u^3 \bigg|_{-1}^1 = \frac{1}{3} [1^3 - (-1)^3] = \frac{2}{3} ;
\]

substituting \( u = \cos \theta \), reversing order

\[
du = -\sin \theta \, d\theta \quad \text{of integration}
\]

(continued)
\[ \int_0^\pi \sin^2 \theta \, d\theta = \int_0^\pi \frac{1}{2} (1 - \cos 2\theta) \, d\theta = \frac{1}{2} \theta - \sin 2\theta \bigg|_0^\pi \\
= \frac{1}{2} \left[ (\pi - 2\sin 2\pi) - (0 - 2\sin 0) \right] = \frac{1}{2} (\pi - 0 - 0 + 0) = \frac{\pi}{2} ; \]
\[ \int_0^\pi \cos^2 \theta \, d\theta = \int_0^\pi \frac{1}{2} (1 + \cos 2\theta) \, d\theta = \frac{1}{2} \theta + \sin 2\theta \bigg|_0^\pi \\
= \frac{1}{2} \left[ (\pi + 2\sin 2\pi) - (0 + 2\sin 0) \right] = \frac{1}{2} (\pi + 0 - 0 - 0) = \frac{\pi}{2} ; * \]

and \[ \int_0^\pi \sin \theta \cos \theta \, d\theta \rightarrow \int_{-1}^1 u (-du) = 0 \ . ** \]

* A quicker way to evaluate this is to apply the Pythagorean Identity and use the result from the preceding integral:
\[ \int_0^\pi \cos^2 \theta \, d\theta = \int_0^\pi (1 - \sin^2 \theta) \, d\theta = (\theta) \bigg|_0^\pi - \int_0^\pi \sin^2 \theta \, d\theta = (\pi - 0) - \frac{\pi}{2} = \frac{\pi}{2} . \]

** We could also use the “double-angle” identity for sine to write this integral as
\[ \int_0^\frac{\pi}{2} \sin 2\theta \, d\theta , \] and then note that we are integrating the sine function over a full period (which is \( \pi \) for \( \sin 2\theta \)).

Hence, the line integral along the semi-circle yields
\[ \int_{B\rightarrow A} (3x^2 + 7y) \, dx + (4x + 4y) \, dy = 16 (-12 \cdot \frac{2}{3} - 7 \cdot \frac{\pi}{2} + 4 \cdot \frac{\pi}{2} + 4 \cdot 0) \\
= 16 (-8 - 3 \cdot \frac{\pi}{2}) = -128 - 24\pi \ . \]

We can at last put this together with the value of the line integral along the x-axis to obtain
\[ \int_C (3x^2 + 7y) \, dx + (4x + 4y) \, dy = 128 + (-128 - 24\pi) = -24\pi \ . \]

(B)

Having computed all of this will serve to confirm the result we arrive at by applying Green’s Theorem: since the semi-circle is a “piecewise-smooth”, closed curve, which we are following in the “counter-clockwise” direction, and the integrand functions have continuous first partial derivatives everywhere within the semi-circle (everywhere in the plane, really), then
\[ \int_C \left( \frac{3x^2 + 7y}{P(x, y)} \right) \, dx + \left( \frac{4x + 4y}{Q(x, y)} \right) \, dy = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dA \]

\[ = \iint_D \left( [4 + 0] - [0 + 7] \right) \, dA = \iint_D (-3) \, dA . \]

Because this integrand is just a constant over the entire semicircle, the value of the integral is simply \((-3)\) times the area of the semi-circle, or

\[ \int_C (3x^2 + 7y) \, dx + (4x + 4y) \, dy = (-3)(+1)(\frac{1}{2} \cdot \pi \cdot 4^2) = -24\pi . \]

CCW area of semi-circle

7) The integral \( \iint_D \sqrt{x^2 + y^2} \, dx \, dy \) integrates the function \( \sqrt{x^2 + y^2} \) over the interior and boundary of a circular region (what mathematicians call a “closed disk”) described by \( x^2 + y^2 \leq 4y \). We can rearrange this inequality and “complete the square” to obtain \( x^2 + (y^2 - 4y + 4) \leq 0 + 4 \Rightarrow x^2 + (y - 2)^2 \leq 4 \); this tells us that the circle is centered at \((0, 2)\) and has a radius of \(2\). So we now have a picture of the situation, but the integral is inconvenient to compute using Cartesian coordinates.

We can, however, arrange to make this work easier by making use of polar coordinates instead. Drawing on the transformation equations between polar and rectangular coordinates, \( x^2 + y^2 = r^2 \) and \( y = r\sin \theta \), the original inequality describing this region becomes \( r^2 \leq 4r\sin \theta \Rightarrow r \leq 4\sin \theta \) (a graph of the boundary curve is presented below). The integrand function becomes \( \sqrt{x^2 + y^2} \to r \). The infinitesimal element of area in Cartesian coordinates, \( dx \, dy \), is replaced by the corresponding element in polar coordinates, \( r \, dr \, d\theta \).

What remains is to consider the limits of integration. The radius to be integrated over runs from the origin \((r = 0)\) to the boundary of the region, \( r = 4\sin \theta \). For the angle variable, we must be careful: the circle \( r = 4\sin \theta \), is also a member of the family of rosettes, \( r = a\sin n\theta \) (discussed in Calculus II in Stewart, Chapter 10). When \( n \) is an odd integer, the rosette has \( n \) petals and is traced once completely in the interval \( 0 \leq \theta < \pi \); continuing from \( \pi \leq \theta < 2\pi \) will retrace the full curve. So we will want to carry out the integration only over \( 0 \leq \theta < \pi \). So saying, the given integral is written in polar coordinates as \( \int_0^{4\sin \theta} \int_0^r r \, dr \, d\theta = \int_0^{4\sin \theta} \int_0^{r^2} r \, dr \, d\theta \).

(E)
8) Unlike Problem 6, where the path for the line integral traces a closed curve, allowing us to apply Green's Theorem directly to simplify the computation enormously, the path in this Problem is not closed. (We can use the Theorem anyway, but we'll see how further down.) For now, we will carry out the line integration: because a straight line and a semi-circle are involved once again, we can refer back to the first solution presented for Problem 6 to describe the method of calculation.

For the integral \( \int_C (2xy + 3) \, dx + (x^2 + 1) \, dy \), we are to follow first the "lower" semicircle \( y = -\sqrt{4 - x^2} \) from \( A(-2, 0) \) to \( B(2, 0) \), and then the line segment from \( B \) to \( C(2, 2) \). The path from \( A \) to \( B \) can be treated most readily using polar coordinates: the circle \( r = 2 \) is followed from \( \theta = \pi \) to \( \theta = 2\pi \), so we would use the substitutions \( x = 2 \cos \theta \) \( \Rightarrow \) \( dx = -2 \sin \theta \, d\theta \) and \( y = 2 \sin \theta \) \( \Rightarrow \) \( dy = 2 \cos \theta \, d\theta \) to transform the integral as follows:

\[
\int_{A \rightarrow B} (2xy + 3) \, dx + (x^2 + 1) \, dy
\]

\[
= 2 \int_\pi^{2\pi} (2 \cdot 2 \cos \theta \cdot 2 \sin \theta + 3)(-2 \sin \theta \, d\theta) + ([2 \cos \theta]^2 + 1)(2 \cos \theta \, d\theta)
\]

\[
= 2 \int_\pi^{2\pi} (-8 \cos \theta \sin^2 \theta - 3 \sin \theta + 4 \cos^3 \theta + 2 \cos \theta) \, d\theta
\]

\[
= 2 \int_\pi^{2\pi} (-8 \cos \theta [1 - \cos^2 \theta] - 3 \sin \theta + 4 \cos^3 \theta + 2 \cos \theta) \, d\theta
\]

using the Pythagorean Identity

\[
= 2 \int_\pi^{2\pi} (12 \cos^3 \theta - 3 \sin \theta - 6 \cos \theta) \, d\theta
\]

\[
= 6 \int_\pi^{2\pi} (4 \cos^3 \theta - \sin \theta - 2 \cos \theta) \, d\theta
\]

(continued)
This now leaves us with three integrations to perform in going term-by-term, two of which are simple and the third requires only an observation:

Because the function \( y = \cos^3 \theta \) goes from \( y = -1 \) at \( \theta = \frac{3\pi}{2} \) to \( y = +1 \) at \( \theta = 2\pi \) symmetrically about \( \theta = \frac{3\pi}{2} \), we find that \( \int_\pi^{2\pi} \cos^3 \theta \, d\theta = 0 \); if we had performed the integration instead on the term involving \( \sin^2 \theta \cos \theta \), the \( u \)-substitution would lead to \( \int_\pi^{2\pi} \sin^2 \theta \cos \theta \, d\theta \rightarrow \int_0^0 u^2 \, du = 0 \);

the other two integrals are simply \( \int_\pi \cos \theta \, d\theta = (-\cos \theta)|_\pi^{2\pi} = (-1) - 1 = -2 \)
and \( \int_\pi \sin \theta \, d\theta = (\sin \theta)|_\pi^{2\pi} = 0 - 0 = 0 \).

We therefore conclude that

\[
\int_A^B (2xy + 3) \, dx + (x^2 + 1) \, dy = 6 (4 \cdot 0 - (-12) - 2 \cdot 0) = 12.
\]

Continuing on the line \( x + 2y = 2 \) from \( B(2,0) \) to \( C(-2,2) \), we can consider handling this line integral parametrically. By writing the equation of the line in slope-intercept form, we have \( y = 1 - \frac{1}{2}x \), showing the slope to be \( -\frac{1}{2} \). We may choose either end of the line segment from \( B \) to \( C \) as the point at which \( t = 0 \); if we start from \( B \), and set \( t = 1 \) at \( C \), then the parametric equations for the segment are \( x = 2 - 4t \Rightarrow dx = -4 \, dt \) and \( y = 0 + 2t \Rightarrow dy = 2 \, dt \). (Any other linear parameterization of the segment would also be fine.) The line integral along this path is then

\[
\begin{align*}
\int_B^C (2xy + 3) \, dx + (x^2 + 1) \, dy &\rightarrow \int_0^1 (2 \cdot [2 - 4t] \cdot [2t] + 3) (-4 \, dt) + ([2 - 4t]^2 + 1) (2 \, dt) \\
&= 2 \int_0^1 [ (-2) (3 + 8t - 16t^2) + (5 - 16t + 16t^2) ] \, dt \\
&= 2 \int_0^1 (48t^2 - 32t - 1) \, dt = 2 (16t^3 - 16t^2 - t)|_0^1 \\
&= 2 \cdot [ (16 - 16 - 1) - (0 - 0 - 0) ] = -2.
\end{align*}
\]

We can finally add the results from the two legs of the line integration together to obtain

\[
\int_C (2xy + 3) \, dx + (x^2 + 1) \, dy = (12) + (-2) = 10.
\]

(B)
There is a subtler approach we can take using Green’s Theorem (though I’m not sure how many students might have thought of it without a hint). If we make a closed path $S$ by adding a leg from $C(-2,2)$ back to $A(-2,0)$, the Theorem would show us that

$$\int_S \left( \frac{2xy + 3}{P(x,y)} \right) \, dx + \left( \frac{x^2 + 1}{Q(x,y)} \right) \, dy = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dA = \iint_D (2x - 2x) \, dA = 0.$$  

One nice thing about this result is that we don’t need to know the area of the region enclosed by $S$ to work with this integral. But the even more important aspect is that we now also know that

$$\int_A^B P \, dx + Q \, dy + \int_B^C P \, dx + Q \, dy + \int_C^A P \, dx + Q \, dy = 0$$

implies

$$\int_A^B P \, dx + Q \, dy + \int_B^C P \, dx + Q \, dy = -\int_C^A P \, dx + Q \, dy ;$$

in other words, we can just evaluate the line integral on the right-hand side of this equation, in order to solve the Problem. (The fact that the integrand in the area integral is zero means that the line integral from $A$ to $C$ is “path-independent”.)

What we notice about this new line integral is that it runs straight down the line $x = -2$ from $y = 2$ to $y = 0$. Hence, $x$ has a constant value in the integration and $dx = 0$, reducing the calculation to

$$\int_C^A (2xy + 3) \, dx + (x^2 + 1) \, dy = \int_2^0 (2xy + 3) \cdot 0 + ([-2]^2 + 1) \, dy$$

$$= \int_2^0 5 \, dy = 5y \bigg|_2^0 = 0 - 10 = -10 .$$

The line integral we were initially asked to compute is then given by

$$\int_C (2xy + 3) \, dx + (x^2 + 1) \, dy = -(-10) = 10 .$$

This approach using Green’s Theorem and path-independence could also be applied to Problem 7 in the Spring 2003 final exam, but it doesn’t save much work there.

9) We are asked to the potential function $f(x, y, z)$ which corresponds to the vector field $\mathbf{F} = <6xy^2 + 5z^2 - 6x^2, 6x^2y - 6zy^2, 10xz - 2y^3>$. We could check to see whether such a function exists by testing to see whether $\nabla \times \mathbf{F} = 0$; if this is the case, then the vector field is said to be “conservative” and a potential function such that $\nabla f = \mathbf{F}$ can be found. Since we are simply asked to determine this function, we can safely assume it is there to be found (we would in fact discover that $\text{curl } \mathbf{F} = 0$).

The gradient of the potential function $f$ must have terms which correspond to the components of the vector field $\mathbf{F}$, so we can integrate each of these terms to construct the function we seek:

(continued)
\[ \frac{\partial}{\partial x} f = F_x = 6xy^2 + 5z^2 - 6x^2 \Rightarrow f = \int F_x \, dx = 3x^2y^2 + 5z^2x - 2x^3 + g(y, z) ; \]
\[ \frac{\partial}{\partial y} f = F_y = 6x^2y - 6zy^2 \Rightarrow f = \int F_y \, dy = 3x^2y^2 - 2zy^3 + h(x, z) ; \]
\[ \frac{\partial}{\partial z} f = F_z = 10xz - 2y^3 \Rightarrow f = \int F_z \, dz = 5xz^2 - 2y^3z + j(x, y) . \]

Here \( F_x \) denotes the \( x \)-component of \( F \), rather than a partial derivative with respect to \( x \); the functions \( g \), \( h \), and \( j \) are "arbitrary functions" in each integration which do not depend upon the variable of integration. Once we intercompare our results from the three integrations, we see that the potential function \( f \) or \( F \) is

\[ f(x, y, z) = 3x^2y^2 + 5xz^2 - 2x^3 - 2y^3z . \] (E)

Any one of the above integrations would be sufficient to eliminate all of the given choices except (E).

10) The region of interest in this Problem lies between the horizontal lines \( y = 1 \) and \( y = 3 \), the \( y \)-axis, and is bounded on the right by the hyperbola \( xy = 3 \); we are to treat this as the form of a plate of uniform density and to locate its centroid. We have two choices as to how to carry out the necessary integrations, one with respect to \( x \), the other with respect to \( y \).

Because of the compound character of the enclosed area (see the graph included above), an integration with respect to \( x \) would need to be broken into two parts, one involving the rectangle between the horizontal lines from \( x = 0 \) to \( x = 1 \), and the second portion between the lower horizontal line and the hyperbola from \( x = 1 \) to \( x = 3 \). This would need to be done in evaluating the "mass" of the region and its two moments:
\[ m = \rho \cdot \left[ \int_{0}^{1} (3 - 1) \, dx + \int_{1}^{3} \left( \frac{3}{x} - 1 \right) \, dx \right] = \rho \cdot \left[ 2x \bigg|_{0}^{1} + (3 \ln x - x) \bigg|_{1}^{3} \right] \]

\[ = \rho \cdot \left[ (2 - 0) + \{(3 \ln 3 - 3) - (3 \ln 1 - 1)\} \right] \]
\[ = \rho \cdot (2 + 3 \ln 3 - 3 - 0 + 1) = \rho \cdot (3 \ln 3) ; \]

\[ M_y = \rho \cdot \left[ \int_{0}^{1} x (3 - 1) \, dx + \int_{1}^{3} x \left( \frac{3}{x} - 1 \right) \, dx \right] = \rho \cdot \left[ \int_{0}^{1} (2x) \, dx + \int_{1}^{3} (3 - x) \, dx \right] \]

\[ = \rho \cdot \left[ x^2 \bigg|_{0}^{1} + (3x - \frac{1}{2} x^2) \bigg|_{1}^{3} \right] = \rho \cdot \left[ (1 - 0) + \{(9 - \frac{9}{2}) - (3 - \frac{1}{2})\} \right] = \rho \cdot 3 ; \]

\[ M_x = \rho \cdot \left[ \int_{0}^{3} \frac{1}{2} [(3)^2 - (1)^2] \, dx + \int_{1}^{3} \frac{1}{2} \left( \frac{3}{x} \right)^2 - (1)^2 \right] \, dx \]

is the difference of the square of each function which must be used

\[ = \frac{1}{2} \rho \cdot \left[ \int_{0}^{3} (9 - 1) \, dx + \int_{1}^{3} \left( \frac{9}{x^2} - 1 \right) \, dx \right] = \frac{1}{2} \rho \cdot \left[ 8x \bigg|_{0}^{1} + (-\frac{9}{x} - x) \bigg|_{1}^{3} \right] \]
\[ = \frac{1}{2} \rho \cdot \left[ (8 - 0) + \{(-3 - 3) - (-9 - 1)\} \right] = \rho \cdot 6 . \]

The coordinates of the centroid for this region are then found from

\[ \bar{x} = \frac{M_y}{m} = \frac{\rho \cdot 3}{\rho \cdot (3 \ln 3)} = \frac{1}{\ln 3} \quad \text{and} \quad \bar{y} = \frac{M_x}{m} = \frac{\rho \cdot 6}{\rho \cdot (3 \ln 3)} = \frac{2}{\ln 3} . \quad \text{(B)} \]

If we integrate this region in the \( y \)-direction instead, the boundaries can be treated in a simpler fashion: we can look at this as a region bounded “above” by a single curve, the hyperbola \( x = \frac{3}{y} \), and “below” by the \( y \)-axis, \( x = 0 \), and carry the integration in one pass from \( y = 1 \) to \( y = 3 \);

\[ m = \rho \int_{1}^{3} \left( \frac{3}{y} \right) \, dy = \rho \cdot (3 \ln y) \bigg|_{1}^{3} = \rho \cdot (3 \ln 3 - 3 \ln 1) = \rho \cdot (3 \ln 3) ; \]

\[ M_X = \rho \int_{1}^{3} y \left( \frac{3}{y} \right) \, dy = \rho \int_{1}^{3} (3) \, dy = \rho \cdot (3y) \bigg|_{1}^{3} = \rho \cdot (9 - 3) = \rho \cdot 6 ; \]

\[ M_Y = \rho \int_{1}^{3} \left( \frac{3}{y} \right)^2 \, dy = \frac{1}{2} \rho \int_{1}^{3} \left( \frac{9}{y^2} \right) \, dy = \frac{1}{2} \rho \cdot (-\frac{9}{y}) \bigg|_{1}^{3} = \frac{1}{2} \rho \cdot (-3 - [-9]) = \rho \cdot 3 , \]

the corresponding moment integrals are now swapped

which leads to the same values as above, but from far less complicated calculations.
11) The rate of change of a multivariate function along a direction \( \mathbf{u} \) from a reference point \( \mathbf{x}_0 \) is called the “directional derivative” \( D_{\mathbf{u}} f \big|_{\mathbf{x}_0} \). The gradient of the function at \( \mathbf{x}_0 \) is the directional derivative in the direction of the greatest rate of increase. It is conventional to use a unit vector \( \hat{\mathbf{u}} \) to indicate the direction for the calculation, so the directional derivative can be evaluated as \( D_{\hat{\mathbf{u}}} f \big|_{\mathbf{x}_0} = \nabla f \cdot \hat{\mathbf{u}} \); this expression applies for any number of variables (dimensions).

In this Problem, we work in three-dimensional space, so the gradient of the function \( \phi(x, y, z) = x^2 + 3xy + xyz^2 \) is given by

\[
\nabla \phi = \left( \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z} \right) = \langle 2x + 3y + yz^2, 3x + xz^2, 2xyz \rangle.
\]

At the reference point \( P(x_0, y_0, z_0) = (1, 1, -1) \), the greatest rate of increase [part (c)] is then given by

\[
\left| \nabla \phi \right|_P = \left| \langle 2x + 3y + yz^2, 3x + xz^2, 2xyz \rangle \right|_{(1,1,-1)} = \left| \langle 6, 4, -2 \rangle \right| = \sqrt{6^2 + 4^2 + (-2)^2} = \sqrt{56} \approx 7.483.
\]

The unit vector in the direction of the gradient (and of most rapid increase) [part (b)] is

\[
\hat{\mathbf{v}} = \left( \frac{\nabla \phi}{\left| \nabla \phi \right|_P} \right)_P = \frac{\langle 6, 4, -2 \rangle}{2\sqrt{14}} = \frac{\sqrt{14}}{14} \langle 3, 2, -1 \rangle.
\]

In the direction from point \( P \) to the point \( Q(2, 2, -3) \), the unit vector is

\[
\hat{\mathbf{u}} = \frac{\langle (2-1), (2-1), ((-3)-(-1)) \rangle}{\sqrt{(2-1)^2 + (2-1)^2 + ((-3)-(-1))^2}} = \frac{\langle 1, 1, -2 \rangle}{\sqrt{1^2 + 1^2 + (-2)^2}} = \frac{\sqrt{6}}{6} \langle 1, 1, -2 \rangle.
\]

The directional derivative of \( \phi \) at \( P \) in this direction can be thought of as the scalar projection of the gradient of \( \phi \) at \( P \) onto \( \hat{\mathbf{u}} \); hence,

\[
D_{\hat{\mathbf{u}}} \phi \big|_P = \hat{\mathbf{v}} \cdot \hat{\mathbf{u}} = \langle 6, 4, -2 \rangle \cdot \frac{\sqrt{6}}{6} \langle 1, 1, -2 \rangle = \frac{\sqrt{6}}{6} (6 \cdot 1 + 4 \cdot 1 + (-2)(-2)) = \frac{\sqrt{6}}{6} \cdot 14 = \frac{7\sqrt{6}}{3} \approx 5.715 \quad \text{[part (a)]}
\]
12) To find the point (or points) on a surface in three-dimensional space that is closest to a chosen point (in this Problem, the origin), we wish to minimize the distance function, \( s = \sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2} \), subject to the constraint that the point(s) \( (x, y, z) \) must be on the surface in question. It should be noted that since distance can only be positive or zero, we can minimize the function \( s^2 \), in order to avoid having to deal with the radical.

For functions of more than one variable, we search for maxima and minima of functions by using the technique of “Lagrange multipliers”: for the function \( f \), subject to a constraining function \( g \), we investigate the components of the gradient vectors in the equation \( \nabla f = \lambda \nabla g \), where \( \lambda \) is an unspecified constant that we make wish to find values for or replace by substitution. We seek to minimize
\[
\begin{align*}
f(x, y, z) &= s^2 = (x - 0)^2 + (y - 0)^2 + (z - 0)^2, \\
\end{align*}
\]
subject to \( g(x, y, z) = z^2 - xy - 32 \) (we have treated the equation for the surface, \( z^2 = xy + 32 \), as a “level surface” for the function \( z^2 - xy \)). The “Lagrange equation” is then \( <2x, 2y, 2z> = \lambda < -y, -x, 2z> \), from which we obtain the three component equations,
\[
(A) : 2x = -\lambda y \quad ; \quad (B) : 2y = -\lambda x \quad ; \quad (C) : 2z = \lambda \cdot 2z.
\]
Equation (C) can only be true for \( \lambda = 1 \), immediately reducing equations (A) and (B) to \( 2x = -y \) and \( 2y = -x \). Inserting (A) into (B) then leads to \( 2(-2x) = -x \), which can only be the case if \( x = 0 \); hence, \( y \) is also zero. Placing these values into the equation for the surface, we find \( z^2 = 0 \cdot 0 + 32 \Rightarrow z = \pm \sqrt{32} \). So we have located two points on the surface, \( (0, 0, -\sqrt{32}) \) and \( (0, 0, \sqrt{32}) \).

To confirm that these are indeed the closest points on the surface to the origin, we can re-write the distance function as a function of two variables, specifically for points on the surface, as \( f(x, y) = s^2 = x^2 + y^2 + (xy + 32) \), and classifying its critical points. Setting the first partial derivatives equal to zero gives us \( f_x = 2x + y = 0 \Rightarrow 2x = -y \) and \( f_y = 2y + x = 0 \Rightarrow 2y = -x \), which verify our results from the Lagrange equation. We now want the second partial derivatives of \( f \), with which to compute the index \( D = f_{xx} f_{yy} - f_{xy}^2 \); these are \( f_{xx} = 2 \), \( f_{yy} = 2 \), and \( f_{xy} = 1 \), yielding \( D = 2 \cdot 2 - 1^2 = 3 \). Since \( D > 0 \) and \( f_{xy} > 0 \), the critical points we have determined above do lie at the minimum distance from the origin.

13) The flux for a vector field \( \mathbf{F} \) through a closed surface \( S \) in three dimensions is the integral of the dot product of \( \mathbf{F} \) with the unit normal vector \( \mathbf{\hat{n}} \) at each infinitesimal bit of surface area (the so-called “areal vector”) summed over the entire surface,
\[
\iint_{S} \mathbf{F} \cdot \mathbf{\hat{n}} \, dS.
\]
A positive value for this integral indicates that the field \( \mathbf{F} \) produces a net flux outward through the surface \( S \) from its interior.

In practice, this integral can become difficult to compute, depending upon the complexity of the vector field and the nature of the boundary surface. It will often simplify the flux calculation to apply the Divergence Theorem,
\[ \Phi = \iiint S \cdot \hat{n} \, dS = \iiint V (\nabla \cdot \vec{F}) \, dV , \]

with the volume integration being carried out through the volume enclosed by \( S \). For the vector field in this Problem, \( \vec{F} = (x^2 + y^2 + z^2) \begin{pmatrix} \frac{x}{M^*} \\ \frac{y}{N^*} \\ \frac{-4z}{P^*} \end{pmatrix} \) (the components are "starred" as a reminder that the multiplying factor must be included when we compute the divergence), we find

\[
\nabla \cdot \vec{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}
\]

\[
= \frac{\partial}{\partial x} [x(x^2 + y^2 + z^2)] + \frac{\partial}{\partial y} [y(x^2 + y^2 + z^2)] + \frac{\partial}{\partial z} [(-4z)(x^2 + y^2 + z^2)]
\]

\[
= 1(x^2 + y^2 + z^2) + x(2x) + 1(x^2 + y^2 + z^2) + y(2y) + (-4)(x^2 + y^2 + z^2) + (-4z)(2z)
\]

by application of Product Rule

\[
= 2x^2 + 2y^2 - 8z^2 - 2(x^2 + y^2 + z^2) = -10z^2
\]

The surface through which the flux passes is bounded by the \( xy \)-plane \((z = 0)\) and the paraboloid \( z = 4 - x^2 - y^2 \); this surface and the enclosed volume are symmetrical about the \( z \)-axis (a so-called "axial symmetry"), so we can carry out our calculation in cylindrical coordinates. In that coordinate system, \( x^2 + y^2 = r^2 \) and the infinitesimal volume element is \( r \, dr \, d\theta \, dz \). So the paraboloid is described by \( z = 4 - r^2 \), indicating that it intersects the \( xy \)-plane at \( 4 - r^2 = 0 \) \( \Rightarrow r = 2 \). The limits of integration are then \( 0 \leq z \leq 4 - r^2, 0 \leq r \leq 2 \), and \( 0 \leq \theta < 2\pi \), and our flux integral becomes

\[
\Phi = \int_0^{2\pi} \int_0^2 \int_0^{4 - r^2} (-10z^2) \, dz \, r \, dr \, d\theta = \int_0^{2\pi} d\theta \int_0^2 \left[ -\frac{10}{3} r^3 \right]_0^{4 - r^2} \, r \, dr
\]

the integrand has no dependence on \( \theta \), so the \( \theta \) integral may be separated out

\[
= 2\pi \int_0^2 -\frac{10}{3} (4 - r^2)^3 \, r \, dr \to -\frac{20\pi}{3} \int_0^4 u^3 (-\frac{1}{2} \, du)
\]

substitute \( u = 4 - r^2 \Rightarrow du = -2r \, dr \)

\[
= -\frac{10\pi}{3} \int_0^4 u^3 \, du = -\frac{10\pi}{3} \left[ \frac{1}{4} u^4 \right]_0^4 = -\frac{10\pi}{3} (64 - 0) = -\frac{640\pi}{3}
\]

The negative value tells us that the net flux of this vector field is into the volume \( V \) of the paraboloid.
For a solid object which requires its density to be described as a function of position within it, $\delta(x, y, z)$, the mass of the object must be computed through the use of a triple integral $m = \iiint_V \delta(x, y, z) \, dV = \iiint_V \delta(x, y, z) \, dx \, dy \, dz$. The volume we are to integrate over in this problem is a sphere with a conical sector removed which extends from the center to the surface of the sphere. The density function is dependent only upon the $z$-coordinate. The form of the density function and the presence of the conical boundary on the volume suggest the use of cylindrical coordinates for the integration, but that will make work with the sphere awkward. Instead, we will set up the volume integration in spherical coordinates.

The original solid sphere can be described simply as $x^2 + y^2 + z^2 = a^2$, so the limits of integration for the radial coordinate are given by $0 \leq \rho \leq a$. Because the conical and spherical surfaces are symmetric about the $z$-axis, and the density function does not depend on the azimuthal angle $\theta$ (which is what initially suggested using cylindrical coordinates), the integration with respect to this variable can be separated out; the interval for integration is $0 \leq \theta < 2\pi$. The excavated conical region has the surface $z = -\sqrt{x^2 + y^2} \rightarrow z = -r$, with $r$ having the usual meaning in polar and cylindrical coordinates. This means that the remaining portion of the sphere is covered in polar angle $\phi$ from its “north pole” on the positive $z$-axis down to that conical surface which is located at

$$\cos \phi = \frac{z}{\rho} = \frac{z}{\sqrt{r^2 + z^2}} = \frac{-r}{\sqrt{r^2 + [-r]^2}} = \frac{-r}{\sqrt{2}r^2} = -\frac{1}{\sqrt{2}} \Rightarrow \phi = \frac{3\pi}{4};$$

the interval of integration is thus $0 \leq \phi < \frac{3\pi}{4}$. [Note: other text books, and other fields such as physics, make $\theta$ the polar angle and $\phi$ the azimuthal angle.]

The infinitesimal volume element in spherical coordinates is $\rho^2 \sin \phi \, d\rho \, d\theta \, d\phi$. The density function within the drilled-out sphere is $\delta(x, y, z) = 1 + z^2 \Rightarrow \delta(\rho, \phi) = 1 + (\rho \cos \phi)^2$. At last, the mass of the object can be computed from the volume integral

$$m = \int_0^{2\pi} \int_0^a \int_0^{3\pi/4} [1 + \rho^2 \cos^2 \phi] \rho^2 \sin \phi \, d\phi \, d\rho \, d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^a \int_0^{3\pi/4} [\rho^2 \sin \phi + \rho^4 \cos^2 \phi \sin \phi] \, d\phi \, d\rho.$$

The azimuthal angle integration is independent of the rest, and so can be separated out. The two polar angle integrals yield

$$\int_0^{3\pi/4} \rho^2 \sin \phi \, d\phi = \rho^2 (-\cos \phi)\bigg|_0^{3\pi/4} = \rho^2 \left([-\frac{\sqrt{2}}{2}] - [-1]\right) = (1 + \frac{\sqrt{2}}{2}) \rho^2$$

and
\[
\int_0^{3\pi/4} \rho^4 \cos^2 \phi \sin \phi \, d\phi \quad \rightarrow \quad \rho^4 \int_{-\sqrt{2}/2}^{\sqrt{2}/2} u^2 \, (-du) = \rho^4 \left. \left( \frac{1}{3} u^3 \right) \right|_{-\sqrt{2}/2}^{\sqrt{2}/2}
\]
subs. \( u = \cos \phi \) , \( du = -\sin \phi \, d\phi \)

\[
= \frac{1}{3} \rho^4 \left[ 1^3 - \left( -\frac{\sqrt{2}}{2} \right)^3 \right] = \frac{1}{3} \left( 1 + \frac{\sqrt{2}}{4} \right) \rho^4 .
\]

Inserting these results into the integral for the mass of the object gives us

\[
m = 2\pi \int_0^a \left[ (1 + \frac{\sqrt{2}}{2}) \rho^2 + \frac{1}{3} (1 + \frac{\sqrt{2}}{4}) \rho^4 \right] \, d\rho
\]

\[
= 2\pi \left[ \frac{1}{3} (1 + \frac{\sqrt{2}}{2}) \rho^3 + \frac{1}{15} (1 + \frac{\sqrt{2}}{4}) \rho^5 \right] \bigg|_0^a = \frac{2\pi}{3} a^3 \left[ (1 + \frac{\sqrt{2}}{2}) + \frac{1}{3} (1 + \frac{\sqrt{2}}{4}) a^2 \right] .
\]

15) This Problem is analogous to the preceding one: we have a cylinder upon which a “skin” of metal is applied, which has a surface density \( \sigma (x, y, z) \); we are asked to find the mass of material deposited onto the wall of the cylinder. Since this plating of material forms what we can treat as a two-dimensional (cylindrical) surface, the mass can be computed by using a surface integral \( m = \iint_S \sigma(x, y, z) \, dA \). The cylindrical form immediately suggests that we should use cylindrical coordinates; however, since the surface density function is given in terms of Cartesian coordinates, we will need to transform that function’s expression.

We have to do a bit of adaptation since the axis of symmetry of this cylinder lies on the \( y \)-axis; thus, this will be one of the directions of integration, carried out from \( y = 0 \) to \( y = 2 \) . The circular cross-section of the cylinder is \( x^2 + z^2 = 16 \), which is centered on the origin and has a radius of \( r = 4 \), so we will revise the usual polar coordinates to \( x = 4 \cos \theta \), \( z = 4 \sin \theta \). In applying cylindrical coordinates to the surface, there is no “thickness”, so we may omit \( dr \) in using the infinitesimal volume element; this leaves us with an infinitesimal surface area element for our cylinder of \( r \, d\theta \, dy = 4 \, d\theta \, dy \). The surface density function can now be modified as \( \sigma (x, y, z) = 1 + yz^2 \rightarrow \sigma (r, y) = 1 + y (4 \sin \theta)^2 \). Finally, we may write the surface integral for the mass of “metal” skin as

\[
m = \int_0^2 \int_0^{2\pi} \left[ 1 + 16 y \sin^2 \theta \right] (4 \, d\theta \, dy)
\]

\[
= 4 \int_0^2 \int_0^{2\pi} \left[ 1 + 16 y \cdot \frac{1}{2} (1 - \cos 2\theta) \right] \, d\theta \, dy
\]

using \( \cos 2\theta = 1 - 2 \sin^2 \theta \)

\[
= 4 \int_0^2 \int_0^{2\pi} (1 + 8y - 8y \cos 2\theta) \, d\theta \, dy = 4 \int_0^2 \left\{ \left[ (1 + 8y) \theta - 4y \sin 2\theta \right] \bigg|_0^{2\pi} \right\} \, dy
\]

(continued)
\[4 \int_0^2 (1+8y) dy = 8\pi \int_0^2 (1+8y) dy\]

\[= 8\pi (y + 4y^2) \bigg|_0^2 = 8\pi (2 + 4 \cdot 2^2) = 144\pi .\]

This calculation can, of course, also be carried out in Cartesian coordinates, but then we will need to bring in the factor for the “tilt” of tangent planes to the cylinder relative to the rectangular cross-section of the cylinder in the \(xz\)-plane. For the half-cylinder above this plane, we use \(z = \sqrt{16 - x^2} \); to find the surface area of the entire cylinder, we can integrate over half of the cross-section, which extends from \(x = 0\) to \(x = 4\) and \(y = 0\) to \(y = 2\), double the result for the surface area of the half-cylinder above the \(xz\)-plane, and double it again for the complete cylinder. The “tangent-plane” factor is

\[\sqrt{\left(\frac{dz}{dx}\right)^2 + \left(\frac{dz}{dy}\right)^2} + 1 = \sqrt{\left(\frac{-x}{\sqrt{16-x^2}}\right)^2 + 0^2 + 1 = \sqrt{1 + \frac{x^2}{16-x^2}} = \frac{4}{\sqrt{16-x^2}} .\]

The surface area of our cylinder is then found from

\[S = 2 \cdot 2 \int_0^4 \int_0^{\frac{4}{\sqrt{16-x^2}}} \frac{4}{\sqrt{16-x^2}} dx dy = 4 \int_0^4 \left[\sin^{-1}\left(\frac{x}{4}\right)\right] dy = 16 \int_0^4 \left(\frac{\pi}{2}\right) dy\]

\[= 16\left(\frac{\pi}{2}\right) \bigg|_0^4 = 16 \cdot \frac{\pi}{2} \cdot 2 = 16\pi ;\]

this agrees with the result for the formula for the “wall” of a cylinder (indeed, this is how the formula is derived through calculus), \(S = 2\pi rh = 2\pi \cdot 4 \cdot 2 = 16\pi .\)

The mass of metal deposited on the surface of the cylinder is now determined from \[m = \iiint_S \sigma(x,y,z) \sqrt{\left(\frac{dz}{dx}\right)^2 + \left(\frac{dz}{dy}\right)^2} + 1 \ dx \ dy .\] The density function is symmetrical about the \(xz\)-plane (since it contains a \(z^2\) factor) and has no dependence on \(x\). So we can use exactly the same limits of integration as we did to calculate the surface area of the cylinder. We take \(z^2 = 16 - x^2\) from the equation of the circular cross-section to write the mass integral as

\[m = 2 \cdot 2 \int_0^4 \int_0^{\sqrt{16-x^2}} (1 + y [16-x^2]) \left(\frac{4}{\sqrt{16-x^2}}\right) dx dy\]

\[= 4 \int_0^4 \int_0^{\frac{4}{\sqrt{16-x^2}}} \left(\frac{4}{\sqrt{16-x^2}}\right) + \left(\frac{4y [16-x^2]}{\sqrt{16-x^2}}\right) dx dy .\]

(continued)
The first term gives the surface area integral we already worked out; the second term leads to

\[
16 \int_0^4 \int_0^2 y \sqrt{16 - x^2} \, dx \, dy = 16 \int_0^2 dy \int_0^4 \sqrt{16 - x^2} \, dx
\]

\[
= 16 \left( \frac{1}{2} \right)^2 \left[ \frac{x}{2} \sqrt{16 - x^2} + \frac{16}{2} \sin^{-1} \left( \frac{x}{4} \right) \right]_0^4
\]

\[
= 16 \left( \frac{1}{2} \cdot 2^2 \right) \cdot \left[ \frac{4}{2} \sqrt{16 - 4^2} + \frac{16}{2} \sin^{-1} \left( \frac{4}{4} \right) \right] - \left[ 0 \cdot \sqrt{16 - 0^2} + \frac{16}{2} \sin^{-1} \left( \frac{0}{4} \right) \right]
\]

\[
= 16 (2) \cdot (0 + 8 \cdot \frac{\pi}{2} - 0 - 0) = 128\pi.
\]

So we also find by this method that \( m = 16\pi + 128\pi = 144\pi \).

16) We are asked to calculate the net work on an object, under the influence of a vector force field \( \mathbf{F} \), around a closed path, but here the curve does not lie in a plane, but extends into three dimensions. If we were working in two dimensions, the line integral for the net work could also be computed using a surface integral under Green’s Theorem. When we wish to make a similar integration in three dimensions, we can apply the extension of that Theorem, known as Stokes’ Theorem,

\[
\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S},
\]

with \( S \) being the surface bounded by the curve \( C \), and \( d\mathbf{S} \) is the normal vector to each infinitesimal “patch” of surface area (the “areal” vector). [Stokes’ Theorem can, in fact, be generalized to any number of dimensions, but that requires a generalization of the definition of vector product as well, which is beyond what is covered in this course.]

The vector field for the force in this Problem,

\[
\mathbf{F} = < x \cos x, 5xy^3 + y \sin y, zy^2 + ze^z >,
\]

is one for which the application of Stokes’ Theorem will be particularly suitable. The closed path over which the force is to be integrated runs from the point \( (2, 0, 0) \) to the point \( (0, 2, 0) \) along a quarter-circle of radius 2 centered on the origin, then to \( (0, 0, 2) \) along a similar quarter-circle, and returns to \( (2, 0, 0) \) on a third such quarter-circle. This suggests the use of polar coordinates in the planes of each leg of the path, but this will make the functions in the components of \( \mathbf{F} \) rather difficult to integrate.

Instead, we will first compute the curl of \( \mathbf{F} \) in preparation for calculating the “Stokes integral”:
\[
\n\nabla \times \mathbf{F} = \begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
x \cos x & 5xy^3 + y \sin y & zy^2 + ze^z
\end{vmatrix}
\]

\[
= \begin{pmatrix}
2yz - 0, -(0 - 0), 5y^3 - 0
\end{pmatrix} = \begin{pmatrix}
2yz, 0, 5y^3
\end{pmatrix} ;
\]

because the curl is not zero, the force \( \mathbf{F} \) is not conservative and so the work around the path \( \mathbf{C} \) will not be zero.

* this is not a definition of the curl operation, but simply a convenient way to describe the computations involved.

We will now work out the surface integral \[ \iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} \], with \( S \) being the surface of the sphere \( x^2 + y^2 + z^2 = 4 \) in the first octant. We can simplify this a bit by integrating in the projection of that surface onto the \( xy \)-plane, which is the interior and boundary of the quarter-circle \( x^2 + y^2 = 4 \), which we will call \( D \). Ordinarily, we would need a “tangent plane” factor (as we described in Problem 15 above), which we obtain by writing an expression for the sphere's surface as a function \( g(x, y) = z = \sqrt{4 - x^2 - y^2} \). However, for this surface integral, the tangent plane factor cancels out, requiring us only to compute

\[
\iint_D \begin{pmatrix}
2yz, 0, 5y^3
\end{pmatrix} \cdot \begin{pmatrix}
-\frac{\partial z}{\partial x}, -\frac{\partial z}{\partial y}, 1
\end{pmatrix} dA
\]

\[
= \iint_D \begin{pmatrix}
2yz, 0, 5y^3
\end{pmatrix} \cdot \begin{pmatrix}
\frac{2x}{\sqrt{4 - x^2 - y^2}}, \frac{2y}{\sqrt{4 - x^2 - y^2}}, 1
\end{pmatrix} dA
\]

\[
= \iint_D \frac{4xyz}{\sqrt{4 - x^2 - y^2}} + 0 + 5y^3 \ dA .
\]

We wish to integrate over the quarter-circle in the \( xy \)-plane; now polar coordinates will be convenient for doing this. We have \( x = r \cos \theta \), \( y = r \sin \theta \), the infinitesimal area element is \( r \, dr \, d\theta \), and the limits of integration are \( 0 \leq r \leq 2 \) and \( 0 \leq \theta \leq \pi/2 \). We get to make one other simplification: since we are integrating in the \( xy \)-plane, \( z = 0 \), reducing the integral to

(continued)
\[
\int \int_D 5y^3 \, dA \rightarrow \int_0^{\pi/2} \int_0^{\pi/2} 5 (r \sin \theta)^3 \, d\theta \, r \, dr = 5 \int_0^2 r^4 \, dr \int_0^{\pi/2} \sin^3 \theta \, d\theta
\]

\[
= 5 \left( \frac{1}{3} r^5 \right) \bigg|_0^2 \cdot \int_0^{\pi/2} (1 - \cos^2 \theta) \sin \theta \, d\theta
\]

using the Pythagorean Identity

\[
= 5 \left( \frac{1}{3} \cdot 2^5 \right) \cdot \int_0^{\pi/2} \sin \theta - \cos^2 \theta \sin \theta \, d\theta
\]

\[
= 32 \left[ (-\cos \theta) \bigg|_0^{\pi/2} - \int_1^0 u^2 \, du \right] = 32 \left[ (-\cos \theta) \bigg|_0^{\pi/2} + \left( \frac{1}{3} u^3 \right) \bigg|_1^1 \right]
\]

using the substitution \( u = \cos \theta \), \( du = \sin \theta \, d\theta \)

\[
= 32 \left[ (0 - [-1]) + \left( \frac{1}{3} \cdot 1^3 - 0 \right) \right]
\]

\[
= 32 \left( 1 + \frac{1}{3} \right) = \frac{128}{3}.
\]

G. Ruffa – 7/11