1) The curvature at a point along a space curve \( r(t) \) depends upon the rate at which the tangent vector to the curve is changing, so we will have a couple of levels of calculation to make. Since the formula for calculating the curvature is provided in a reference section at the end of the exam booklet, we will be able to obtain the required result by finding the first two derivatives of the vector function \( r(t) \):

\[
\vec{r}(t) = \left< 2t^2 + 2, \frac{8}{3}t^{3/2} + \frac{4}{3}, 2t + 2 \right>
\]

\[
\Rightarrow \vec{r}'(t) = \frac{d}{dt} \vec{r}(t) = \left< 4t, 4t^{1/2}, 2 \right>
\]

\[
\Rightarrow \vec{r}''(t) = \frac{d}{dt} \vec{r}'(t) = \left< 4, 2t^{-1/2}, 0 \right> ;
\]

\[
\vec{r}'(t) \times \vec{r}''(t) \overset{\text{**}}{=} \begin{vmatrix}
   i & j & k \\
   4t & 4t^{1/2} & 2 \\
   4 & 2t^{-1/2} & 0
\end{vmatrix} = \left< -4t^{-1/2}, 8, -8t^{1/2} \right> ;
\]

* keep in mind that this determinant is not the definition for the cross product, but simply a convenience for calculation

\[
|\vec{r}'(t)| = \sqrt{(4t)^2 + (4t^{1/2})^2 + 2^2} = \sqrt{16t^2 + 16t + 4}
\]

\[
= 2\sqrt{4t^2 + 4t + 1} = 2\sqrt{(2t + 1)^2} = 2(2t + 1) ,
\]

\[
|\vec{r}'(t) \times \vec{r}''(t)| = \sqrt{(-4t^{-1/2})^2 + 8^2 + (-8t^{1/2})^2}
\]

\[
= \sqrt{\frac{16}{t} + 64 + 64t} = 4\sqrt{\frac{4t + 4 + \frac{1}{t}}{t}}
\]

\[
= 4\sqrt{(2\sqrt{t} + \frac{1}{\sqrt{t}})^2} = 4(2\sqrt{t} + \frac{1}{\sqrt{t}}) ;
\]

\[
\kappa(t) = \frac{|\vec{r}'(t) \times \vec{r}''(t)|}{|\vec{r}'(t)|^3} = \frac{4(2\sqrt{t} + \frac{1}{\sqrt{t}})}{2^3(2t + 1)^3} = \frac{2\sqrt{t} + \frac{1}{\sqrt{t}}}{2(2t + 1)^3}
\]

\[
\kappa(1) = \frac{2\sqrt{1} + \frac{1}{\sqrt{1}}}{2(2 \cdot 1 + 1)^3} = \frac{2 + 1}{2(2 + 1)^3} = \frac{1}{18} .
\]
We could, of course, have begun the evaluation of vectors at \( t = 1 \) immediately after determining the derivatives of \( r(t) \); thus,

\[
\vec{r}'(1) = \begin{pmatrix} 4 \cdot 4^{1/2} \cdot 2 \end{pmatrix} = \begin{pmatrix} 4 \cdot 4 \cdot 2 \end{pmatrix} \Rightarrow |\vec{r}'(1)| = \sqrt{4^2 + 4^2 + 2^2} = 6 ;
\]

\[
\vec{r}''(1) = \begin{pmatrix} 4 \cdot 2 \cdot 1^{-1/2} \cdot 0 \end{pmatrix} = \begin{pmatrix} 4 \cdot 2 \cdot 0 \end{pmatrix} ;
\]

\[
\vec{r}'(1) \times \vec{r}''(1) = \begin{vmatrix}
i & j & k \\
4 & 4 & 2 \\
4 & 2 & 0
\end{vmatrix} = \begin{pmatrix} -4 & 8 & -8 \end{pmatrix}
\]

\[
\Rightarrow |\vec{r}'(1) \times \vec{r}''(1)| = \sqrt{(-4)^2 + 8^2 + (-8)^2} = 12 ;
\]

\[
\kappa(1) = \frac{|\vec{r}'(1) \times \vec{r}''(1)|}{|\vec{r}'(1)|^3} = \frac{12}{6^3} = \frac{1}{18} .
\]

We could also proceed from the basic definition of curvature, \( \kappa(t_0) = \frac{dT}{ds} \bigg|_{t=t_0} \), in order to write

\[
\mathbf{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|} = \begin{pmatrix} 4t \cdot 4t^{1/2} \cdot 2 \end{pmatrix} = \begin{pmatrix} 2t \cdot 2t^{1/2} \cdot 1 \end{pmatrix} \frac{2(2t+1)}{2t+1}
\]

\[
\Rightarrow \frac{d\mathbf{T}}{dt} = \frac{d}{dt} \begin{pmatrix} 2t \cdot 2t^{1/2} \cdot 1 \\
2t+1 \cdot 2t^{1/2} \cdot 1 \\
2t+1 \cdot 2t^{1/2} \cdot 1 
\end{pmatrix} = \begin{pmatrix} \frac{2}{(2t+1)^2} \cdot \frac{(1-2t) \sqrt{t}}{(2t+1)^2} \cdot \frac{-2}{(2t+1)^2} \end{pmatrix} = \frac{1}{(2t+1)^3} \begin{pmatrix} 2 \cdot \frac{1-2t}{\sqrt{t}} \cdot -2 \end{pmatrix}
\]

\[
\Rightarrow \left| \frac{d\mathbf{T}}{dt} \right| = \frac{1}{(2t+1)^2} \sqrt{2^2 + \left( \frac{1-2t}{\sqrt{t}} \right)^2 + (-2)^2} = \frac{1}{(2t+1)^2} \sqrt{\frac{9 - 4t + 4t^2}{t}} ;
\]

\[
\kappa(t) = \frac{\left| \frac{d\mathbf{T}}{ds} \right|}{ds/dt} = \frac{\left| \frac{d\mathbf{T}}{dt} \right|}{|\vec{r}'(t)|} = \frac{1}{(2t+1)^3} \sqrt{\frac{9 - 4t + 4t^2}{t}} = \sqrt{\frac{9 - 4t + 4t^2}{2(2t+1)^3}}
\]

\[
\Rightarrow \kappa(1) = \sqrt{\frac{1}{2(2+1)^3}} = \sqrt{\frac{9}{2(3)^3}} = \frac{1}{18} .
\]
2) The second partial derivative \( f_{xy}(x, y) \) is equivalent to writing \( \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial x} \right) \); that is to say, the order of partial differentiation is read in the subscripts from left to right. So we have

\[
f(x, y) = xe^{xy^2}
\]

\[
\Rightarrow f_x(x, y) = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(xe^{xy^2}) = \left( \frac{\partial}{\partial x} x \right) \cdot (e^{xy^2}) + x \cdot \left( \frac{\partial}{\partial x} e^{xy^2} \right)
\]

\[
= 1(e^{xy^2}) + x\left(y^2e^{xy^2}\right) = (1 + xy^2)e^{xy^2};
\]

\[
\Rightarrow f_{xy}(x, y) = \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial x} \right) = \frac{\partial}{\partial y}[ (1 + xy^2)e^{xy^2} ]
\]

\[
= \left( \frac{\partial}{\partial y} [1 + xy^2] \right) \cdot (e^{xy^2}) + (1 + xy^2) \cdot \left( \frac{\partial}{\partial y} e^{xy^2} \right)
\]

\[
= (0 + x \cdot 2y)(e^{xy^2}) + (1 + xy^2) \cdot \left( 2xye^{xy^2} \right)
\]

\[
= (2xy + 2xy + 2x^2y^3)e^{xy^2} = (4xy + 2x^2y^3)e^{xy^2} \quad \text{(C)}
\]

3) The equation of a plane with a normal vector \( \mathbf{n} = <a, b, c> \) is given by the vector equation \( \mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = a(x-x_0) + b(y-y_0) + c(z-z_0) = 0 \). For the tangent plane at a point \( (x_0, y_0, z_0) \) on the level surface of a function \( f(x, y, z) = K \), \( K \) being a constant, this normal vector is the gradient \( \nabla f \big|_{(x_0,y_0,z_0)} = \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right) \big|_{(x_0,y_0,z_0)} \) evaluated at that point.

For the level surface \( x^2y + y^2z + xz^2 = 19 \), the gradient vector at a point on that surface is found to be \( \nabla f = <2xy + z^2, x^2 + 2yz, y^2 + 2xz> \). Hence, the gradient at \( (1, -2, 3) \) is \( \nabla f \big|_{(1,-2,3)} = \left\{ 2 \cdot 1 \cdot (-2) + 3^2, 1^2 + 2 \cdot (-2) \cdot 3, (-2)^2 + 2 \cdot 1 \cdot 3 \right\} = <5, -11, 10> \). The tangent plane to our level surface at this point is thus

\[
5(x - 1) + [-11](y - [-2]) + 10(z - 3) = 5x - 5 - 11y - 22 + 10z - 30 = 0
\]

\[
\Rightarrow 5x - 11y + 10z = 57 \quad \text{(A)}
\]
The function \( L(x, y, z) = 5(x - 1) - 11(y + 2) + 10(z - 3) \) also describes the linearization of the function \( f(x, y, z) \) in the neighborhood of the point \( (1, -2, 3) \).

4) We look for critical points of functions of more than variable in a manner similar to what we do for a function of a single variable; the change that is necessary is that we must find the values of the variables for which all of the first partial derivatives are zero simultaneously. For the function \( f(x, y) = x^3 + 2x^2 + 2y^2 - 2xy + 4x - 8y \), we find

\[
\begin{align*}
  f_x &= 3x^2 + 4x - 2y + 4 = 0 \\
  f_y &= 4y - 2x - 8 = 0 
\end{align*}
\]

\[
\Rightarrow 3x^2 + 4x - (x + 4) + 4 = 0
\]

\[
\Rightarrow 3x^2 + 3x = 0 \quad \Rightarrow 3(x + 1) = 0 \quad \Rightarrow x = 0, -1.
\]

So the critical points are located at points where the \( x \)-coordinate has these values.

In order to classify these points for a function of two variables, we need to determine the second partial derivatives of \( f \) and evaluate the index \( D = f_{xx}f_{yy} - (f_{xy})^2 \) at each critical point. The three principal types are:

\[
\begin{align*}
  D > 0, f_{xx} > 0 : & \quad \text{local minimum} \\
  D > 0, f_{xx} < 0 : & \quad \text{local maximum} \\
  D < 0 : & \quad \text{saddle point}
\end{align*}
\]

We find the second partial derivatives to be

\[
\begin{align*}
  f_{xx} &= 6x + 4, \quad f_{yy} = 4, \quad f_{xy} = -2 \quad \Rightarrow D = 4(6x + 4) - (-2)^2 = 24x + 12.
\end{align*}
\]

At the point \( (0, 2) \), we have \( D = 24 \cdot 0 + 12 > 0 \) and \( f_{xx} > 0 \), so this critical point is a local minimum. The other critical point at \( x = -1 \) has \( D = 24 \cdot (-1) + 12 < 0 \), making it a saddle point. (B)

5) In considering how to change the order of integration in the double integral

\[
\int_0^{1/2} dx \int_{x^3}^{x^{1/4}} x^2 y \ dy
\]

we need to interpret what it represents. The integrand function can be taken to be \( f(x, y) = x^2 y \). The limits of integration in the \( y \)-direction then extend from \( y = x^3 \) to \( y = x^{1/4} \), in other words, the integration is carried out between these two curves. The integration in the \( x \)-direction is carried from \( x = 0 \) to \( x = 1/2 \); but, in fact, we find upon investigation the location of possible intersection points between the two curves that \( x^3 = x^{1/4} \Rightarrow x^3 - x^{1/4} = 0 \Rightarrow x(x^2 - 1/4) = 0 \). So there are three intersection points \( x = -1/2 \), \( x = 0 \), and \( x = 1/2 \), meaning that the limits of integration in the \( x \)-direction cover a region between two of these intersection points, making it an enclosed region. A graph of the area of integration is shown below.
For the given double integral, the integration is made in the \( y \)-direction first, with \( y = \frac{1}{4} x \) being the "upper curve" and \( y = x^3 \), the "lower curve". Since we now wish to integrate first in the \( x \)-direction, we need to express these functions in terms of the variable \( y \). The "upper curve" relative to the \( y \)-axis is \( y = x^3 \rightarrow x = \sqrt[3]{y} \) and the "lower curve" is now \( y = \frac{1}{4} x \rightarrow x = 4y \). The limits of integration in the \( y \)-direction are automatically set by the intersection points, so we will carry the integration from \( y = 0 \) to \( y = \frac{1}{8} \). The equivalent double integral is thus

\[
\int_{1/8}^{\sqrt[3]{y}} \int_{4y}^{x^2} y \, dx \, dy \quad .
\]

(B)

6) The idea in this Problem is similar to that of the previous one, except that we need to consider a figure in three dimensions. The elliptic paraboloid \( z = x^2 + 2y^2 \) opens \( \textit{upward} \) from the origin \( (0, 0, 0) \), having the parabolic cross-sections \( z = x^2 \) in the \( xz \)-plane (shown in the graph below) and \( z = 2y^2 \) in the \( yz \)-plane; its cross-sections parallel to the \( xy \)-plane are ellipses centered on the \( z \)-axis with equations \( x^2 + 2y^2 = c \), with \( c \) being a positive number representing the "height" above the \( xy \)-plane. The parabolic cylinder \( z = 10 - x^2 \) opens \( \textit{downward} \) from the \textit{line} given by \( x = 0, z = 10 \), so all of its cross-sections parallel to the \( yz \)-plane are downward-opening parabolas.
The volume enclosed by these two surfaces is a bit complicated to describe, but we can say something about where they intersect. If we set the two functions equal, we find \( x^2 + 2y^2 = 10 - x^2 \Rightarrow 2x^2 + 2y^2 = 10 \Rightarrow x^2 + y^2 = 5 \). This suggests that the two surfaces meet on a circle, which is what the curve looks like seen from directly above the origin, but it is in fact a space curve which is warped upwards and downwards. Nevertheless, the equation we have found is useful in determining limits of integration for the enclosed volume. The parabolic cylinder produces a “roof” for the volume above this curve, and the elliptic paraboloid provides a “floor”. So in the \( z \)-direction, we will integrate upward from \( z = x^2 + 2y^2 \) to \( z = 10 - x^2 \) (this already eliminates all of the choices except (B) and (E)).

The circle \( x^2 + y^2 = 5 \) then provides the “footprint” over which we need to integrate in the \( y \) - and \( x \)-directions. If we solve this equation for \( y \), we obtain \( y = \pm \sqrt{5 - x^2} \), which represent two semi-circles, one below and one above the \( x \)-axis. The domain of these two functions is \( -\sqrt{5} \leq x \leq \sqrt{5} \). Hence, we will need to integrate in the \( y \)-direction from the “lower” semi-circle, \( y = -\sqrt{5 - x^2} \), to the “upper” semi-circle, \( y = \sqrt{5 - x^2} \), in the interval \( -\sqrt{5} \leq x \leq \sqrt{5} \). Consequently, the integration of \( f(x,y,z) \) over the volume for this three-dimensional figure can be written as

\[
\frac{1}{2} \int_{-\sqrt{5}}^{\sqrt{5}} \int_{-\sqrt{5 - x^2}}^{\sqrt{5 - x^2}} \frac{1}{5} \left( 5 - x^2 - y^2 \right) dy \, dx
\]
\[ \int_{-\sqrt{5}}^{\sqrt{5}} \int_{-\sqrt{5-x^2}}^{\sqrt{5-x^2}} \int_{x^2+2y^2}^{10-x^2} f(x,y,z) \, dz \, dy \, dx \quad . \] (E)

7) We again have a volume integral for a figure which is enclosed by two surfaces. One is the infinite cylinder \( x^2 + y^2 = 4 \), which at any “height” \( z \) above or below the \( xy \)-plane has a cross-section of a circle of radius 2 centered on the \( z \)-axis. The second surface is that of the cone \( x^2 + y^2 = z^2 \), which at any “height” has a circular cross-section of radius \( \sqrt{z^2} = |z| \) centered on the \( z \)-axis (this means that the cone extends both above and below the \( xy \)-plane, forming two “nappes”).

Since this enclosed volume is symmetrical about the \( xy \)-plane, we will integrate over the positive \( z \)-direction, and then double the resulting volume. We can see that the two surfaces intersect where \( z^2 = x^2 + y^2 = 4 \) \( \Rightarrow \) \( z = \pm 2 \), so the limits of integration for our volume-to-be-doubled are \( z = 0 \) to \( z = 2 \). Because the cross-sections of the volume are circular, it will be convenient to work with cylindrical coordinates, which are polar coordinates for these cross-sections parallel to the \( xy \)-plane. Each cross-section is then bounded on the “outside” by \( x^2 + y^2 = r^2 = 4 \) \( \Rightarrow \) \( r = 2 \), and on the “inside” by \( x^2 + y^2 = r^2 = z^2 \) \( \Rightarrow \) \( r = z \) (that is, each cross-section is an “annulus” or ring). Therefore, the integration in the radial (\( r \)-) direction has the limits \( r = z \) to \( r = 2 \). This integration works the same way in any direction away from the \( z \)-axis, so it has no dependence on the azimuthal direction, \( \theta \); consequently, the azimuthal integration can be carried out independently over the interval \( \theta = 0 \) to \( \theta = 2\pi \). The infinitesimal volume element in cylindrical coordinates is \( dV = r \, dr \, d\theta \, dz \). The volume integration for the complete enclosed figure (both “upper” and “lower” halves) can then be written and evaluated as

\[
V = 2 \int_0^{2\pi} \int_0^2 \int_0^z r \, dr \, d\theta \, dz = 2 \int_0^{2\pi} \int_0^2 r \, dr \, dz
\]

\[
= 2 \cdot \theta \int_0^{2\pi} \left( \frac{1}{2} r^2 \right) \bigg|_0^2 \, dz = 2 \cdot 2\pi \cdot \int_0^2 \left( \frac{1}{2} \cdot 2^2 - \frac{1}{2} \cdot z^2 \right) \, dz
\]

\[
= 4\pi \cdot \left( 2z - \frac{1}{6} z^3 \right) \bigg|_0^2 = 4\pi \left[ (2 \cdot 2 - \frac{1}{6} \cdot 2^3) - (0 - 0) \right]
\]

\[
= 4\pi \left( 4 - \frac{4}{3} \right) = 4\pi \left( \frac{12 - 4}{3} \right) = \frac{32\pi}{3} \quad . \] (A)

8) In Calculus II, we showed that arclength \( s \) along a curve \( y = f(x) \) from \( x = a \) to \( x = b \) could be found by integrating \( s = \int_a^b \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \, dx \). We can write an analogous expression in three dimensions for the surface area of a figure defined by
\[ z = f(x, y) \text{ over a closed region } D \text{ in the } xy\text{-plane, } S = \iint_D \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2} \, dA. \]

The three-dimensional surface in question is given by \( z = 2 \sqrt{xy} \), for which we find

\[ f_x = 2 \frac{\partial}{\partial x} (xy)^{1/2} = 2 \cdot \frac{1}{2} \cdot (xy)^{-1/2} \cdot \frac{\partial}{\partial x} (xy) = \frac{y}{\sqrt{xy}}, \]

\[ f_y = 2 \frac{\partial}{\partial y} (xy)^{1/2} = 2 \cdot \frac{1}{2} \cdot (xy)^{-1/2} \cdot \frac{\partial}{\partial y} (xy) = \frac{x}{\sqrt{xy}}. \]

\[ \Rightarrow \sqrt{1 + \left(\frac{y}{\sqrt{xy}}\right)^2 + \left(\frac{x}{\sqrt{xy}}\right)^2} = \sqrt{1 + \frac{y}{\sqrt{xy}} + \frac{x}{\sqrt{xy}}} = \frac{\sqrt{xy + x^2 + y^2}}{xy}. \]

The region \( D \) in the \( xy\)-plane is a triangle with vertices \((0, 0)\), \((2, 0)\), and \((0, 1)\), so it lies in the first quadrant and is bounded by the \( x\)- and \( y\)-axes and the line \( y = 1 - \frac{1}{2}x \). We wish to integrate in the \( y\)-direction from the \( x\)-axis \((y = 0)\) up to the line we have identified, and we will integrate in the \( x\)-direction from the \( y\)-axis \((x = 0)\) to the \( x\)-intercept of the oblique line, at \( x = 2 \). Bringing all of these results together, our surface area integral is then given by

\[ S = \int_0^2 \int_0^{2 - \frac{1}{2}x} \sqrt{\frac{x^2 + y^2 + xy}{xy}} \, dy \, dx. \]

**B**

9) We can take on this line integral in one of two ways. The direct approach is to make an integration from \( A(-1, 0) \) to \( B(1, 0) \) along the \( x\)-axis, with \( y = 0 \) and \( dy = 0 \), followed by an integration from point \( B \) back to point \( A \) along the semi-circular path, using polar coordinates. The two segments tracing the closed curve \( C \) thus yield

\[ \int_C (12x + 4y) \, dx + (7x - 2y) \, dy = \]

\[ \int_{A \rightarrow B} (12x + 4y) \, dx + (7x - 2y) \, dy + \int_{B \rightarrow A} (12x + 4y) \, dx + (7x - 2y) \, dy \]

\[ = \int_{-1}^1 (12x + 4 \cdot 0) \, dx + (7x - 2y) \cdot 0 \]

\[ + \int_0^\pi (12 \cdot \cos \theta + 4 \cdot \sin \theta) (-\sin \theta \, d\theta) + (7 \cdot \cos \theta - 2 \cdot \sin \theta) (\cos \theta \, d\theta) \]

using \( x = \cos \theta, \, dx = -\sin \theta \, d\theta; \, y = \sin \theta, \, dy = \cos \theta \, d\theta \)

and integrating from the positive \((\theta = 0)\) to the negative \( x\)-direction \((\theta = \pi)\)
\[ = \int_{-1}^{1} 12x \, dx + \int_{0}^{\pi} (7 \cos^2 \theta - 4 \sin^2 \theta - 14 \sin \theta \cos \theta) \, d\theta \]

\[ = (6x^2)|_{-1}^{1} + \int_{0}^{\pi} \left( 7 \cdot \frac{1}{2} [1 + \cos 2\theta] - 4 \cdot \frac{1}{2} [1 - \cos 2\theta] - 7 \cdot 2\theta \right) \, d\theta \]

\[ = (6 \cdot 1^2 - 6 \cdot [-1]^2) + \int_{0}^{\pi} \left( \frac{3}{2} + \frac{11}{2} \cos 2\theta - 7 \cdot 2\theta \right) \, d\theta \]

\[ = 0 + \left( \frac{3}{2} \pi - \frac{11}{4} \sin 2\pi + \frac{7}{2} \cos 2\pi \right) \left|_{0}^{\pi} \right. \]

\[ = \left( \frac{3}{2} \pi - \frac{11}{4} \sin 2\pi + \frac{7}{2} \cos 2\pi \right) - \left( \frac{3}{2} \cdot 0 - \frac{11}{4} \sin 0 + \frac{7}{2} \cos 0 \right) \]

\[ = \frac{3}{2} \pi - 0 + \frac{7}{2} - 0 + 0 - \frac{7}{2} = \frac{3\pi}{2} . \quad (C) \]

This first is the approach we would take using the “tools” of Calculus II. However, we are now also familiar with Green’s Theorem, which states that for a region \( D \) bounded by a simple closed curve \( C \) which is piecewise-continuous and positive-oriented (that is, we trace it in the “counter-clockwise” direction), if \( P(x, y) \) and \( Q(x, y) \) are functions with continuous partial derivatives in an open region containing \( D \), then

\[ \int_{C} P(x, y) \, dx + Q(x, y) \, dy = \iint_{D} \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dA . \]

Using this method, our line integral becomes

\[ \int_{C} (12x + 4y) \, dx + (7x - 2y) \, dy \]

\[ = \iint_{D} \left( \frac{\partial}{\partial x} [7x - 2y] - \frac{\partial}{\partial y} [12x + 4y] \right) \, dA = \iint_{D} (7 - 4) \, dA = 3 \iint_{D} dA . \]

The integrand in the Green's Function integral reduces to a constant, so the line integral is simply equal to \( 3 \) times the area of the enclosed semicircle. Since this semi-circle has a radius of \( 1 \), we obtain

\[ \int_{C} (12x + 4y) \, dx + (7x - 2y) \, dy = 3 \cdot \frac{1}{2} \pi \cdot 1^2 = \frac{3\pi}{2} . \]
10) This Problem only requires a straightforward calculation of the curl of the vector function (the function which describes the vector field) 

\[ F = \langle 6x + 3y^2z - yz^2, 3xy - 3y^2z, yz^2 - 4xz \rangle. \]

The 3×3 determinant used here is not the definition of this vector operation, but serves as a convenience for remembering the order of differentiations:

\[
\tilde{\nabla} \times \tilde{F} = \begin{vmatrix}
    \mathbf{i} & \mathbf{j} & \mathbf{k} \\
    \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
    6x + 3y^2z - yz^2 & 3xy - 3y^2z & yz^2 - 4xz
\end{vmatrix}
\]

\[ = \left\{ (z^2 - 0) - (0 - 3y^2), -(0 - 3z) + (0 + 3y^2 - 2yz), (3y - 0) - (0 + 6yz - z^2) \right\}
\]

\[ = \left\{ 3y^2 + z^2, 3y^2 - 2yz + 4z, 3y - 6yz + z^2 \right\}. \quad (D) \]

11) A vector field \( \mathbf{F} \) is said to be conservative when its line integral around a simple closed curve is zero (\( \oint_{C} \mathbf{F} \cdot d\mathbf{r} = 0 \)); this is equivalent to saying that its line integral between two points is “path-independent”, and that the curl of the vector field is zero (the vector field is then also called “irrotational” or “curl-free”). For the vector field \( \mathbf{F} = \langle \alpha xy + \beta y^2 + 3x^2, 6x^2 + 2xy - y^2 \rangle \), which is only two-dimensional, the analogue of the curl calculation is "\( \tilde{\nabla} \times \tilde{F} \)" = \( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \), with \( \mathbf{F} = \langle P, Q \rangle \) having components which are functions of \( x \) and \( y \). Thus, our vector field is conservative if

\[
\frac{\partial}{\partial x} (6x^2 + 2xy - y^2) - \frac{\partial}{\partial y} (\alpha xy + \beta y^2 + 3x^2) = 0
\]

\[ \Rightarrow (12x + 2y) - (\alpha x + 2\beta y) = 0. \]

This will hold true if we can match \( x \)- and \( y \)-terms, giving us \( 12x = \alpha x \) and \( 2y = 2\beta y \). Hence, this vector field is conservative in the case where \( \alpha = 12 \) and \( \beta = 1 \).

(E)

12) The integration in this Problem can also be taken on in one of two ways. The vector field involved is \( \mathbf{F} = \langle y, -x, xy \rangle \); in the double integral \( \iint_{S} (\tilde{\nabla} \times \tilde{F}) \cdot d\mathbf{S} \), the surface to be integrated over is the section of \( z = g(x, y) = xy \) which “lies above” the square with vertices \( O(0, 0, 0) \), \( P(0, 1, 0) \), \( Q(1, 1, 0) \), and \( R(1, 0, 0) \)
The normal vector that is chosen for \( dS \) points “downward” from the surface toward the \( xy \)-plane, giving the boundary square \( C \) a “clockwise” (negative) orientation. The calculation of the curl gives us

\[
\nabla \times \vec{F} = \begin{vmatrix}
  \hat{i} & \hat{j} & \hat{k} \\
  \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
  y & -x & xy
\end{vmatrix}
= \langle (x - 0) , (-y + 0) , (-1 - 1) \rangle = \langle x , -y , -2 \rangle.
\]

The vector expression for the “areal vector” on the surface \( S \) is

\[
d\vec{S} = \hat{n} \, dS = \langle -g_x , -g_y , 1 \rangle \, dA = \langle -y , -x , 1 \rangle \, dA ,
\]

where \( dA \) is the infinitesimal area element in the projection of the surface onto the \( xy \)-plane and the factor \( \sqrt{1+g_x^2+g_y^2} \) cancels out in the denominator for the unit vector \( \hat{n} \) against the factor in \( dS = \sqrt{1+g_x^2+g_y^2} \, dA \). We must reverse this, however, because the statement of the Problem asks that we use the “downward” orientation, which is in the direction opposite to \( \hat{n} \). Our double integral is then

\[
\vec{\int_S} (\nabla \times \vec{F}) \cdot d\vec{S} = \vec{\int_S} \langle x , -y , -2 \rangle \cdot [\langle -y , -x , 1 \rangle] \, dA
= \vec{\int_S} (xy - xy + 2) \, dA = \vec{\int_S} 2 \, dA = 2 \int_S dA .
\]

Because the constant factor may be extracted from the integration, the value of the double integral is simply 2 times the area of the square lying in the \( xy \)-plane. This is a unit square (square of side 1), so our integral equals 2.

(A)

Since \( S \) is an oriented piecewise-continuous surface bounded by a simple closed curve (the unit square) and \( F \) has continuous partial derivatives in an open region containing \( S \) (since \( g(x,y) \) and \( F \) include products of simple, continuous functions), we could also invoke Stokes’ Theorem, which states that

\[
\vec{\int_S} (\nabla \times \vec{F}) \cdot d\vec{S} = \int_C \vec{F} \cdot d\vec{r} .
\]

So we can replace the surface integration with a line integral around the unit square in the \( xy \)-plane. Because the choice of \( dS \) is “downward”, we must follow the square around in a clockwise path from \( O \) to \( P \) to \( Q \) to \( R \) to \( O \) (which is actually why I labeled the points as I did at the beginning). This line integration has four parts, but they are all fairly easy, since the sides of the square are parallel to either the \( x \)- or \( y \)-axis:

\[
\int_C \vec{F} \cdot d\vec{r} = \int_{O\to P} \langle y , -x , xy \rangle \cdot \langle 0 , dy , 0 \rangle + \int_{P\to Q} \langle y , -x , xy \rangle \cdot \langle dx , 0 , 0 \rangle
+ \int_{Q\to R} \langle y , -x , xy \rangle \cdot \langle 0 , dy , 0 \rangle + \int_{R\to O} \langle y , -x , xy \rangle \cdot \langle dx , 0 , 0 \rangle
\]
\[ = \int_{0}^{1} -x \, dy + \int_{0}^{1} y \, dx + \int_{1}^{0} -x \, dy + \int_{1}^{0} y \, dx \]

\[
= 0 + \int_{0}^{1} 1 \, dx + \int_{1}^{0} -1 \, dy + 0 = x\bigg|_{0}^{1} - y\bigg|_{0}^{1} = 1 - (-1) = 2 .
\]

13) This Problem revisits a topic we discussed in Calculus II. The infinitesimal arclength element in the plane is \( ds = \sqrt{dx^2 + dy^2} \), which is always what we are integrating in order to find the distance between two points measured along a curve. When the curve is described by \( y = f(x) \) and we will be integrating along the \( x \)-direction from \( x = a \) to \( x = b \), then we can factor \( dx \) out of the radical to calculate the arclength as

\[
s = \int_{A}^{B} ds = \int_{a}^{b} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx .
\]

For our function, we have

\[
y = \frac{1}{4} x^2 - \frac{1}{2} \ln x \implies \frac{dy}{dx} = \frac{1}{2} x - \frac{1}{2} \cdot \frac{1}{x}
\]

\[
\implies \left(\frac{dy}{dx}\right)^2 = \left(\frac{1}{2} x - \frac{1}{2} \cdot \frac{1}{x}\right)^2 = \frac{1}{4} x^2 - \frac{1}{2} + \frac{1}{4x^2}
\]

\[
\implies 1 + \left(\frac{dy}{dx}\right)^2 = \frac{1}{4} x^2 + \frac{1}{2} + \frac{1}{4x^2} = \left(\frac{1}{2} x + \frac{1}{2x}\right)^2
\]

\[
s = \int_{1}^{e} \sqrt{\left(\frac{1}{2} x + \frac{1}{2x}\right)^2} \, dx = \int_{1}^{e} \frac{1}{2} x + \frac{1}{2x} \, dx
\]

\[
= \left(\frac{1}{4} x^2 + \frac{1}{2} \ln x\right)\bigg|_{1}^{e} = \left(\frac{1}{4} e^2 + \frac{1}{2} \ln e\right) - \left(\frac{1}{4} \cdot 1^2 + \frac{1}{2} \ln 1\right)
\]

\[
= \frac{1}{4} e^2 + \frac{1}{2} - \frac{1}{4} - 0 = \frac{1}{4} e^2 + \frac{1}{4} \text{ or } \frac{1}{4} (e^2 + 1) .
\]

14) We have a function of two variables \( z = F(x, y) \), for which \( x \) and \( y \) are themselves functions of two variables, \( x = g(r, \theta) \) and \( y = h(r, \theta) \). In order to calculate first partial derivatives such as \( \frac{\partial z}{\partial r} \), we need the multi-variate version of the Chain Rule. So we will be working with
Thus, the extended version of the Chain Rule gives us
\[
\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial r}
\]
and
\[
\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial \theta}.
\]

For this Problem, we find
\[
z = \ln \left( x + \sqrt{x^2 + y^2} \right), \quad x = r \cos \theta, \quad y = r \sin \theta
\]
\[
\Rightarrow \frac{\partial z}{\partial x} = \frac{\partial}{\partial x} \left[ \ln \left( x + \sqrt{x^2 + y^2} \right) \right] = \frac{1}{x + \sqrt{x^2 + y^2}} \cdot \frac{\partial}{\partial x} \left( x + \sqrt{x^2 + y^2} \right)
\]
\[
= \frac{1}{x + \sqrt{x^2 + y^2}} \cdot \left( 1 + \frac{x}{\sqrt{x^2 + y^2}} \cdot 2x \right),
\]
\[
\frac{\partial z}{\partial y} = \frac{\partial}{\partial y} \left[ \ln \left( x + \sqrt{x^2 + y^2} \right) \right] = \frac{1}{x + \sqrt{x^2 + y^2}} \cdot \frac{\partial}{\partial y} \left( x + \sqrt{x^2 + y^2} \right)
\]
\[
= \frac{1}{x + \sqrt{x^2 + y^2}} \cdot \left( 1 + \frac{x}{\sqrt{x^2 + y^2}} \cdot 2y \right) = \frac{1}{x + \sqrt{x^2 + y^2}} \cdot \left( \frac{y}{\sqrt{x^2 + y^2}} \right);
\]
\[
\frac{\partial x}{\partial r} = \cos \theta, \quad \frac{\partial y}{\partial r} = \sin \theta, \quad \frac{\partial x}{\partial \theta} = -r \sin \theta, \quad \frac{\partial y}{\partial \theta} = r \cos \theta.
\]

Thus, the extended version of the Chain Rule gives us
\[
\frac{\partial z}{\partial r} = \left[ \frac{1}{x + \sqrt{x^2 + y^2}} \cdot \left( 1 + \frac{x}{\sqrt{x^2 + y^2}} \right) \right] \cos \theta + \left[ \frac{1}{x + \sqrt{x^2 + y^2}} \cdot \left( \frac{y}{\sqrt{x^2 + y^2}} \right) \right] \sin \theta
\]
\[
= \left[ \frac{1}{r \cos \theta + r} \cdot \left( 1 + \frac{r \cos \theta}{r} \right) \right] \cos \theta + \left[ \frac{1}{r \cos \theta + r} \cdot \left( \frac{r \sin \theta}{r} \right) \right] \sin \theta
\]
\[
= \left( \frac{1 + \cos \theta}{r \cos \theta + r} \right) \cos \theta + \left( \frac{\sin \theta}{r \cos \theta + r} \right) \sin \theta
\]
\[
= \cos \theta + \cos^2 \theta + \sin^2 \theta \quad \text{using rectangular to polar transformations}
\]
\[
= \cos \theta + \frac{1}{r \cos \theta + r} = \cos \theta + \frac{1}{\cos \theta + 1} = \frac{1}{r}
\]
and
\[
\frac{\partial z}{\partial \theta} = \left[ \frac{1}{x + \sqrt{x^2 + y^2}} \cdot \left( 1 + \frac{x}{\sqrt{x^2 + y^2}} \right) \right] (-r \sin \theta) + \left[ \frac{1}{x + \sqrt{x^2 + y^2}} \cdot \left( \frac{y}{\sqrt{x^2 + y^2}} \right) \right] (r \cos \theta)
\]
\[
= \left( \frac{1 + \cos \theta}{r \cos \theta + 1} \right) (-r \sin \theta) + \left( \frac{\sin \theta}{r \cos \theta + 1} \right) (r \cos \theta)
\]
\[
= -\frac{\sin \theta - \sin \theta \cos \theta}{\cos \theta + 1} + \frac{\sin \theta \cos \theta}{\cos \theta + 1} = \frac{\sin \theta}{\cos \theta + 1}.
\]

15) The method of Lagrange multipliers is an extension of the approach we take in finding local extrema of a function of a single variable. Rather than simply find the locations (if any) of horizontal tangents to a curve \( y = f(x) \), we look for places on the surface of a multivariate function \( f(x_1, \ldots, x_n) \) where the normal vector is aligned with the normal vector of a constraint function \( g(x_1, \ldots, x_n) \). Since these normal vectors are given by the gradients of each function, we can attempt to solve the vector equation \( \nabla f = \lambda \nabla g \), where \( \lambda \) is a constant real number (hence, one gradient is just a scalar multiple of the other one). Each component of the gradient vectors gives us one of \( n \) equations to be solved simultaneously (if possible) for the coordinates of critical points.

We are seeking in this Problem the maximal value of the function \( f(x, y, z) = xy^2z^3 \), subject to the requirements that \( g(x, y, z) = x + y + z = 12 \) and that \( x, y, \) and \( z \) all be positive. (This last condition is rather important: the function \( f \) can still be positive if \( x \) and \( z \) are both negative, but then \( y \) could become increasing positive as the other variables become increasingly negative, all without limit, which would allow \( f \) to grow without limit. With all three variables being positive, we are also guaranteed that we will have found the maximum value for \( f \) in that octant.) The two gradients we will work with are

\[
\nabla f = \left\{ \frac{\partial}{\partial x} xy^2z^3, \frac{\partial}{\partial y} xy^2z^3, \frac{\partial}{\partial z} xy^2z^3 \right\} = \left\{ y^2z^3, 2xyz^3, 3xy^2z^2 \right\} \quad \text{and} \quad \nabla g = \left\{ \frac{\partial}{\partial x} [x + y + z], \frac{\partial}{\partial y} [x + y + z], \frac{\partial}{\partial z} [x + y + z] \right\} = \left\{ 1, 1, 1 \right\}.
\]

Applying the Lagrange multiplier equation, we obtain the three component equations

\[
y^2z^3 = \lambda \cdot 1, \quad 2xyz^3 = \lambda \cdot 1, \quad 3xy^2z^2 = \lambda \cdot 1.
\]

In this case, all three components of \( \nabla f \) are equal to \( \lambda \) and thus to each other at critical points*, so we can "pair up" the expressions for these components in three separate equations:

\[
y^2z^3 = 2xyz^3 \quad \Rightarrow \quad y = 2x, \quad y^2z^3 = 3xy^2z^2 \quad \Rightarrow \quad z = 3x,
\]

\[
2xyz^3 = 3xy^2z^2 \quad \Rightarrow \quad 2z = 3y.
\]
we end up eliminating \( \lambda \) in this situation; in other Lagrange multiplier calculations, we may need to determine its value.

This last result is redundant (it follows from the first two), so we will not give it further attention. The first two equations we’ve derived can be inserted into the constraint equation to yield \( x + y + z = x + 2x + 3x = 12 \Rightarrow 6x = 12 \Rightarrow x = 2 \). Hence, we have \( y = 4 \), \( z = 6 \), and the maximal value of \( f \) in the first octant is \( f(2,4,6) = 2 \cdot 4^2 \cdot 6^3 = 6912 \).

16) We are asked to evaluate \( \iint_D \frac{1}{1 + x^2 + y^2} \, dA \), with the region \( D \) being described by \( x^2 + y^2 \leq 1 \) (referred to as the “closed unit disk centered on the origin”). The appearance of the terms \( x^2 + y^2 \) and the region of integration being a circle suggest that polar coordinates might be of value here, and indeed they are. Upon writing \( x^2 + y^2 \) as \( r^2 \) and using the infinitesimal area element in polar coordinates, \( dA = r \, dr \, d\theta \), we have

\[
\iint_D \frac{1}{1 + x^2 + y^2} \, dA \rightarrow \int_0^{2\pi} \int_0^1 \frac{1}{1 + r^2} \, r \, dr \, d\theta = \int_0^{2\pi} \int_0^1 \frac{r}{1 + r^2} \, dr
\]

\[
= \int_0^{2\pi} \left( \int_0^1 \frac{1}{2u} \, du \right) \left( \ln \left( \frac{1}{2u} \right) \right) \left|_1^2 \right. = \pi \ln 2.
\]

using \( u = 1 + r^2 \), \( du = 2r \, dr \)

17) This is a Problem similar to Problem 6 above. We again have a paraboloid \( z = x^2 + y^2 \) opening upward from the origin and also the oblique plane \( z = 2x \). Since the paraboloid extends upward without limit, it must be the case that the volume enclosed by these two surfaces has the paraboloid for its “floor” and the plane as its “roof”. If we set the two surface equations equal to one another, we obtain

\[
x^2 + y^2 = 2x \quad \Rightarrow \quad x^2 - 2x + y^2 = 0 \quad \Rightarrow \quad (x^2 - 2x + 1) \quad \Rightarrow \quad y^2 = 0 + 1 \quad \text{“completing the square”}
\]

\[
\Rightarrow \quad (x - 1)^2 + y^2 = 1,
\]

which is a circle of radius 1 centered at \((1,0)\). This will serve as the “base figure” in the \(xy\)-plane over which we carry out our volume integration (shown in the graph below). Since this circle and the enclosed volume are symmetrical about the \(x\)-axis, we can integrate the half of the volume for which \( x \geq 0 \), and then double the result. The limits of integration in the \(y\)-direction run from the \(x\)-axis (\(y = 0\)) to the “upper” semi-circle \( y = \sqrt{1 - (x - 1)^2} \), with the limits in the \(x\)-direction extending from
\[ x = 0 \text{ to } x = 2 \text{. The height of the enclosed volume reaches from the paraboloid upward to the oblique plane, so we can write the volume integral in Cartesian coordinates as } \]

\[ V = 2 \int_{0}^{2} \int_{0}^{\sqrt{1 - (x - 1)^2}} (2x) - (x^2 + y^2) \ dy \ dx \text{.} \]

That radical in the limits of integration makes this integral a bit daunting to perform, so we may wish to look for a more convenient form in which to express it. We may recall from Calculus II that the circle above can also be described by the \textit{polar} equation \[ r = 2 \cos \theta \text{; the } x^2 + y^2 \text{ terms in the integrand also suggest the possibility of working with polar coordinates. The limits of integration for the “upper semi-circle” would then run in the radial direction from } r = 0 \text{ to } r = 2 \cos \theta \text{ and in angle from } \theta = 0 \text{ to } \theta = \pi/2 \text{. With the usual formulas for transformation from rectangular to polar coordinates and the infinitesimal area element being } dA = r \ dr \ d\theta \text{, this makes our volume integral into} \]

\[ \rightarrow V = 2 \int_{0}^{\pi/2} \int_{0}^{2 \cos \theta} \left( (2r \cos \theta) - (r^2) \right) r \ dr \ d\theta = 2 \int_{0}^{\pi/2} \int_{0}^{2 \cos \theta} \left( 2r^2 \cos \theta - (r^3) \right) \ dr \ d\theta \]

\[ = 2 \int_{0}^{\pi/2} \left( \frac{2}{3} r^3 \cos \theta - \frac{1}{4} r^4 \right)_{0}^{2 \cos \theta} \ d\theta \]

\[ = 2 \int_{0}^{\pi/2} \left( \frac{2}{3} \cdot 8 - \frac{1}{4} \cdot 16 \right) \cos \theta \ d\theta = \frac{8}{3} \int_{0}^{\pi/2} \cos^4 \theta \ d\theta \text{;} \]

it will be helpful here to write \( \cos^4 \theta \) as a sum of terms with multiplied angles, hence,
\[
\cos^4 \theta = (\cos^2 \theta)^2 = \left( \frac{1}{2} \left[ 1 + \cos 2\theta \right] \right)^2 = \frac{1}{4} (1 + 2 \cos 2\theta + \cos^2 2\theta)
\]
\[
= \frac{1}{4} (1 + 2 \cos 2\theta + \left\{ \frac{1}{2} \left[ 1 + \cos 4\theta \right] \right\}) = \frac{1}{4} + \frac{2}{4} \cos 2\theta + \frac{1}{8} + \frac{1}{8} \cos 4\theta
\]
\[
= \frac{3}{8} + \frac{1}{2} \cos 2\theta + \frac{1}{8} \cos 4\theta ; \ \text{and so our volume integral is at last}
\]

\[
V = \frac{8}{3} \int_{0}^{\pi/2} \left( \frac{3}{8} + \frac{1}{2} \cos 2\theta + \frac{1}{8} \cos 4\theta \right) d\theta
\]
\[
= \left[ (\theta + \frac{4}{3} \int \frac{1}{2} \sin 2\theta + \frac{1}{3} \frac{1}{4} \sin 4\theta) \right]_{0}^{\pi/2}
\]
\[
= \left( \frac{\pi}{2} + \frac{2}{3} \sin \pi + \frac{1}{12} \sin 2\pi \right) - (0 + \frac{2}{3} \sin 0 + \frac{1}{12} \sin 0)
\]
\[
= \frac{\pi}{2} + 0 + 0 - 0 - 0 - 0 = \frac{\pi}{2} .
\]

18) Because of the mixture of the variables \(x\) and \(y\) in the ratio in the exponent of the integrand function of \(\iint_{D} e^{x+y} \ dA\), this is a difficult integral to approach directly, all the more so as the region of integration \(D\) is a quadrilateral with two non-parallel sides, being bounded by the \(x\)- and \(y\)-axes and the oblique lines \(x + y = 2\) and \(x + y = 4\). When we encountered such challenges with integration of functions of one variable, we could start by making a variable transformation that might make the integration more tractable.

Here, we will need to transform two variables, so we will use the numerator and denominator of the exponent ratio as a guide and propose \(x = u + v\) and \(y = u - v\). Just as we need to determine how to alter the differential \(dx\) in the one-variable integral into \(du\), we must work out an analogous change for \(dA = dx \ dy\) here. We accomplish this by computing the “Jacobian determinant” \(J = \begin{vmatrix} x_{u} & x_{v} \\ y_{u} & y_{v} \end{vmatrix}\), which allows us to write \(dA = |J| \ du \ dv\). We observe that \(x + y = 2u\) and \(x - y = 2v\) and that \(J = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2\); the boundaries of our region of integration become \(u + v = 0\)

\[\Rightarrow \ v = -u \ (x = 0 , \text{ the } y\text{-axis}) , \ u - v = 0 \Rightarrow \ v = u \ (y = 0 , \text{ the } x\text{-axis}) , \ 2u = 2 \Rightarrow \ u = 1 \ (\text{the line } x + y = 2) , \ \text{and} \ 2u = 4 \Rightarrow \ u = 2 \ (\text{the line } x + y = 4)\].

Graphs of the region in the \(xy\)- and \(uv\)-planes are shown below. [Notice that the direction around the trapezium of the vertices \(A , B , C\), and \(D\) is reversed in the \(uv\)-plane: this is the significance of the Jacobian determinant being negative.]
The transformed double integral will now have limits of \( v = -u \) to \( v = u \) in the \( v \)-direction and \( u = 1 \) to \( u = 2 \) in the \( u \)-direction, making our calculation

\[
\int_1^2 \int_{-u}^u e^{2u} \cdot | -2 | \, dv \, du = 2 \int_1^2 \int_{-u}^u e^v \, dv \, du = 2 \int_1^2 \left( \frac{1}{1/u} \right)^{1/2} e^{1/u} \bigg|_{-u}^u \, du
\]

\[
= 2 \int_{-1}^1 u e^u \bigg|_{-u}^u \, du = 2 \int_{-1}^1 \left( u e^u - u e^{-u} \right) \, du = 2 \int_{-1}^1 u \left( e^x - e^{-x} \right) \, du
\]

\[
= 2 \left( e - e^{-1} \right) \int_{-1}^1 u \, du = 2 \left( e - e^{-1} \right) \cdot \left( \frac{1}{2} \right) \left( 2^2 - 1^2 \right) = 3 \left( e - e^{-1} \right).
\]

19) In evaluating the integral of a function over a surface which is not flat and parallel to one of the coordinate planes, we cannot simply compute, say,

\[
\iint_S f(x,y) \, dA = \iint_S f(x,y) \, dx \, dy,
\]

because we must take into account the inclination of tangent planes at each point on the surface with respect, in this example, to the \( xy \)-plane. Just as we introduce a differential \( \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \, dx \) when integrating arclength along the \( x \)-direction, we must include an analogous “inclination factor” for two dimensions, \( \sqrt{1 + \left( \frac{\partial z}{\partial x} \right)^2 + \left( \frac{\partial z}{\partial y} \right)^2} \), in the surface integral.

In this Problem, the surface \( S \) is the portion of the cone, \( x^2 + y^2 = z^2 \), from \( z = 0 \) to \( z = 1 \). Implicit differentiation of the equation for the cone gives us
\[
\frac{\partial}{\partial x} z^2 = \frac{\partial}{\partial x} x^2 + \frac{\partial}{\partial y} y^2 \quad \Rightarrow \quad 2z \frac{\partial z}{\partial x} = 2x + 0 \quad \Rightarrow \quad \frac{\partial z}{\partial x} = \frac{x}{z} ,
\]
and similarly, \( \frac{\partial z}{\partial y} = \frac{y}{z} . \) So the inclination factor for the cone is
\[
\sqrt{1 + \left(\frac{x}{z}\right)^2 + \left(\frac{y}{z}\right)^2} = \sqrt{1 + \left(\frac{x^2 + y^2}{z^2}\right)} = \sqrt{1 + \left(\frac{z^2}{z^2}\right)} = \sqrt{2}
\]
(which makes sense, since the surface of the cone makes a constant 45º inclination to the \( xy \)-plane), and our surface integral becomes \( \iiint_S z^2 \; dS \rightarrow \iiint_D z^2 \cdot \sqrt{2} \; dA , \) with the region of integration \( D \) being the area in the \( xy \)-plane which is the projection of the cone. Since our surface extends “upward” to \( z = 1 \), this projection will be the circle \( x^2 + y^2 = 1^2 \).

While we could go on to complete the integration in Cartesian coordinates, we may also note that the cone and its projection onto the \( xy \)-plane are symmetrical about the \( z \)-axis. This suggests that we could use polar coordinates instead. We have \( z^2 = x^2 + y^2 = r^2 \); since the infinitesimal area element in polar coordinates is \( dA = r \; dr \; d\theta \), this integral can now be written as
\[
\sqrt{2} \iiint_D z^2 \; dA = \sqrt{2} \iiint_D r^2 \cdot r \; dr \; d\theta = \sqrt{2} \int_0^{2\pi} d\theta \int_0^1 r^3 \; dr
\]
\[
= \sqrt{2} \cdot \theta \bigg|_0^{2\pi} \cdot \left(\frac{1}{4} r^4\right) \bigg|_0^1 = \frac{2\pi \sqrt{2}}{4} = \frac{\pi \sqrt{2}}{2} .
\]

20) In the previous Problem, we calculated a surface integral for a scalar function; now we are determining the surface integral for a vector function. We can write
\[
\iiint_S \vec{F} \cdot d\vec{S} = \iiint_S \vec{F} \cdot \hat{n} \; dS ,
\]
with \( dS \) being the so-called “areal vector” which points in a direction normal to the surface at each point, and \( \hat{n} \) is the unit vector in that normal direction. If we follow the approach described in Section 16.7 of Stewart (6th edition) for vector surface integrals, we can transform this into \( \iiint_D \vec{F} \cdot \hat{n} \; dA \), with \( D \) being the projection of the surface onto, say, the \( xy \)-plane, and \( \hat{n} \) being the normal vector to the surface (not adjusted to unit length).

The surface we are working with is the “downward-opening” paraboloid, \( z = 1 - x^2 - y^2 \); using implicit differentiation in the same manner as we did above, we find \( \frac{\partial z}{\partial x} = \frac{-x}{z} \), \( \frac{\partial z}{\partial y} = \frac{-y}{z} \), so the normal vector pointing “outward” from the paraboloid’s surface is given by \( \hat{n} = \left\{\frac{-\partial z}{\partial x}, \frac{-\partial z}{\partial y}, 1\right\} = \left\{\left(\frac{-x}{z}\right), \left(\frac{-y}{z}\right), 1\right\} \). The
projection of our paraboloid onto the $xy$-plane is the circle $x^2 + y^2 = 1^2$, which will serve as our region of integration $D$, making the surface integral

$$
\iint_S \vec{F} \cdot \hat{n} \, dA = \iint_S \left( \begin{array}{c} x^3 \\ y^3 \\ 1 + z^3 \end{array} \right) \cdot \left( \begin{array}{c} \frac{x}{z} \\ \frac{y}{z} \\ 1 \end{array} \right) \, dA
$$

$$
= \iint_S \left( \frac{x^4}{z} \right) + \left( \frac{y^4}{z} \right) + \left( 1 + z^3 \right) \, dA.
$$

Unfortunately, given the form of the function $z$, this integral is not very convenient to work out in either Cartesian or polar coordinates.

However, for vector surface integrals, we have an alternative approach available to us by applying the Divergence Theorem: if $E$ is a simply-connected volume and $S$ is its boundary surface with positive (“outward”) orientation, then for a vector field $\vec{F}$ with components possessing continuous partial derivatives in an open volume containing $E$, $\iint_S \vec{F} \cdot d\vec{S} = \iiint_E (\nabla \cdot \vec{F}) \, dV$. The divergence of our vector field $\vec{F} = \langle x^3, y^3, 1 + z^3 \rangle$ is given by

$$
\nabla \cdot \vec{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} = \frac{\partial}{\partial x} x^3 + \frac{\partial}{\partial y} y^3 + \frac{\partial}{\partial z} (1 + z^3)
$$

subscripts here are being used to indicate components of $\vec{F}$

$$
= 3x^2 + 3y^2 + 3z^2 = 3x^2 + 3y^2 + 3(1 - x^2 - y^2)^2.
$$

This is the function upon which we will perform a volume integration, with the volume $E$ being the portion of our “downward-opening” paraboloid above the $xy$-plane.

This paraboloid, like the cone in the previous Problem, is symmetrical about the $z$-axis, so it will be straightforward to apply cylindrical coordinates. We will integrate “upward” in the $z$-direction from the $xy$-plane ($z = 0$) to the apex of the paraboloid’s surface at $z = 1$. In the radial direction, we integrate from the $z$-axis ($r = 0$) to the paraboloid’s surface at $z = 1 - (x^2 + y^2) = 1 - r^2 \implies r = \sqrt{1 - z}$. Since none of this depends upon the azimuthal angle $\theta$, we can separately integrate in azimuth once around the $z$-axis. Applying polar coordinates here also, we can write

$$
\nabla \cdot \vec{F} = 3(x^2 + y^2) + 3(1 - (x^2 + y^2))^2 = 3[r^2 + (1 - r^2)^2] = 3(1 - r^2 + r^4).
$$

With the infinitesimal volume element in cylindrical coordinates being $dV = r \, dr \, d\theta \, dz$, our necessary integration becomes

$$
\iiint_S \vec{F} \cdot d\vec{S} = \iiint_E 3(1 - r^2 + r^4) \, r \, dr \, d\theta \, dz = 3 \int_0^{2\pi} d\theta \, \int_0^1 \sqrt{1-z} \, \int_0^1 (r - r^3 + r^5) \, dr \, dz
$$
\[
= 3 \cdot \theta \left|_0^{2\pi} \int_0^1 \left( \frac{1}{2} r^2 - \frac{1}{4} r^4 + \frac{1}{6} r^6 \right) \right|^{\sqrt{1-z}}_0 dz
\]
\[
= 3 \cdot 2\pi \int_0^1 \left( \frac{1}{2}[\sqrt{1-z}]^2 - \frac{1}{4}[\sqrt{1-z}]^4 + \frac{1}{6}[\sqrt{1-z}]^6 - 0 + 0 - 0 \right) dz
\]
\[
= 6\pi \int_0^1 \frac{1}{2}(1-z) - \frac{1}{4}(1-z)^2 + \frac{1}{6}(1-z)^3 \; dz \;
\]
at this point, we could multiply out the binomials and collect terms to produce a single cubic polynomial to integrate, but it will be less trouble to work out if we make the substitution \( w = 1-z \), \( dw = -dz \):
\[
\rightarrow 6\pi \int_0^1 \frac{1}{2}w - \frac{1}{4}w^2 + \frac{1}{6}w^3 \; (-dw) = 6\pi \left( \frac{1}{4}w^2 - \frac{1}{12}w^3 + \frac{1}{24}w^4 \right) \bigg|_0^1
\]
\[
= 6\pi \left( \frac{1}{4} - \frac{1}{12} + \frac{1}{24} - 0 + 0 - 0 \right) = 6\pi \left( \frac{6-2+1}{24} \right) = 6\pi \cdot \frac{5}{24} = \frac{5\pi}{4} .
\]

The integration of the divergence of \( F \) over the volume of the paraboloid can also be handled by “slicing up” the volume along the \( z \)-axis to integrate the divergence in infinitesimal “discs” of volume \( dV = \pi [ r(z) ]^2dz \), thusly:
\[
\int_S \int_E \vec{F} \cdot d\vec{S} = \int_E \int_S 3(1 - r^2 + r^4) \; r \; dr \; d\theta \; dz \rightarrow \int_0^1 3(1 - r^2 + r^4) \cdot \pi r^2 \; dz ,
\]
with \( r(z) = \sqrt{1-z} \)
\[
= 3\pi \int_0^1 (r^2 - r^4 + r^6) \; dz = 3\pi \int_0^1 (1-z) - [1-z]^2 + [1-z]^3 \; dz
\]
\[
\rightarrow 3\pi \int_1^0 w - w^2 + w^3 \; (-dw) = 3\pi \left( \frac{1}{2}w^2 - \frac{1}{3}w^3 + \frac{1}{4}w^4 \right) \bigg|_1^0
\]
\[
= 3\pi \left( \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - 0 + 0 - 0 \right) = 3\pi \left( \frac{6-4+3}{12} \right) = 3\pi \cdot \frac{5}{12} = \frac{5\pi}{4} .
\]