Math 2263
Spring 2003 Final Exam Problems

This exam contains 12 multiple-choice questions, worth 15 points each, and 4 written-answer problems, worth 30 points each, for a total of 300 points.

1. An equation for the tangent plane to the surface \( x^2y - 2y^2z + 3xz^2 = 12 \) at the point \((2, 1, -1)\) is

   A. \( z = 7(x - 2) + 8(y - 1) - 1 \)
   B. \( 7x + 8y - 14z = 0 \)
   C. \( 7x + 8y - 14z = 12 \)
   D. \( 7x + 8y - 14z = 36 \)
   E. \( (2xy + 3z^2)(x - 2) + (x^2 - 4yz)(y - 1) + (-2y^2 + 6xz)(z + 1) = 0 \)

2. For curve \( \mathbf{r}(t) = (e^t + e^{-t})\mathbf{i} + (e^t - e^{-t})\mathbf{j} + 2t \mathbf{k} \), a unit tangent vector at the point corresponding to \( t = 1 \) is

   A. \( \frac{1}{\sqrt{2}}(\mathbf{i} + \mathbf{k}) \)
   B. \( (e + \frac{1}{e})\mathbf{i} + (e - \frac{1}{e})\mathbf{j} + 2\mathbf{k} \)
   C. \( \frac{1}{\sqrt{2(1 + \frac{1}{e^2})}}[(e + \frac{1}{e})\mathbf{i} + (e - \frac{1}{e})\mathbf{j} + 2\mathbf{k}] \)
   D. \( \frac{1}{\sqrt{2(1 + \frac{1}{e^2})}}[(e - \frac{1}{e})\mathbf{i} + (e + \frac{1}{e})\mathbf{j} + 2\mathbf{k}] \)
   E. \( \frac{\sqrt{2}}{e + \frac{1}{e}}[(e + \frac{1}{e})\mathbf{i} + (e - \frac{1}{e})\mathbf{j} + 2\mathbf{k}] \)

3. Suppose \( u = x^3 + y^3 \) and \( x = e^{3t} \) and \( y = e^{4-t} \). Then \( \frac{\partial u}{\partial t} \) at \( s = 2, \ t = 1 \) equals

   A. \( e^6 + e^3 \)
   B. \( 6e^6 - 3e^3 \)
   C. \( 6e^6 + 3e^3 \)
   D. \( 6e^4 - 3e^3 \)
   E. \( 3e^6 - 3e^3 \)
4. Let \( f(x, y) = x^3 + \frac{1}{2}y^2 - 3xy + 9x - y \). Then

A. \( f \) has saddle points at both \((-1, -2)\) and \((-2, -5)\) and no other critical points

B. \( f \) has a local minimum at \((1, 4)\) and a local maximum at \((2, 7)\), and no other critical points

C. \( f \) has a saddle point at \((1, 4)\) and a local minimum at \((2, 7)\), and no other critical points

D. \( f \) has a saddle point at \((1, 4)\) and a local maximum at \((2, 7)\), and no other critical points

E. \( f \) has a local maximum at \((1, 4)\) and a local minimum at \((2, 7)\), and no other critical points

5. Changing the integration order in the iterated integral \( \int_0^2 \int_{\frac{y}{3}}^{\frac{2}{y}} f(x, y)\,dy\,dx \) leads to

A. \( \int_0^1 \int_0^{2y} f(x, y)\,dx\,dy + \int_1^3 \int_0^{\frac{x}{3}} f(x, y)\,dx\,dy \)

B. \( \int_0^3 \int_{\frac{y}{3}}^{\frac{x}{3}} f(x, y)\,dx\,dy \)

C. \( \int_0^2 \int_{\frac{y}{3}}^{\frac{2}{y}} f(x, y)\,dx\,dy \)

D. \( \int_0^{\frac{y}{2}} \int_0^{2y} f(x, y)\,dx\,dy + \int_1^3 \int_{\frac{y}{3}}^{\frac{x}{3}} f(x, y)\,dx\,dy \)

E. \( \int_0^1 \int_0^{\frac{x}{2}} f(x, y)\,dx\,dy + \int_1^3 \int_0^{\frac{x}{3}} f(x, y)\,dy\,dx \)

6. Let \( C \) be the closed curve in \( \mathbb{R}^2 \) consisting of the segment on the \( y \)-axis joining \((0, -1)\) and \((0, 1)\), and of the semicircle \( x^2 + y^2 = 1 \) with \( x \geq 0 \) (also joining \((0, -1)\) and \((0, 1)\)), oriented counterclockwise. Then

\[
\int_C (3x + 8y)\,dx + (3x - 7y)\,dy \quad \text{equals}
\]

A. 0
B. \(-\frac{5}{2}\pi\)
C. \(\frac{5}{2}\pi\)
D. \(-5\pi\)
E. \(5\pi\)
7. Let \( C \) be the curve in \( \mathbb{R}^2 \) which goes from the point \((3, 9)\) to \((2, 4)\) along the parabola \( y = x^2 \) and then continues on to \((0, 0)\) along the line \( y = 2x \). The integral \( \int_C 2xydx + (x^2 + 2)dy \) equals

A. 180
B. -180
C. 99
D. -99
E. 0

8. The solid \( E \) consists of all points \((x, y, z)\) in \( \mathbb{R}^3 \) inside the cylinder \( x^2 + y^2 = 4 \) and outside the cone \( x^2 + y^2 = z^2 \), i.e., \( E = \{(x, y, z) \mid x^2 + y^2 \leq 4 \text{ and } z \leq x^2 + y^2 \} \). Then in polar coordinates its volume is [note: the outline of the cone looks like the symbol “X” viewing horizontally]

A. \( \int_0^{2\pi} \int_0^2 2r^2drd\theta \)
B. \( \int_0^{2\pi} \int_0^4 2r^2drd\theta \)
C. \( \int_0^{2\pi} \int_0^2 r^2drd\theta \)
D. \( \int_0^{2\pi} \int_0^2 2rdrd\theta \)
E. \( \int_0^{2\pi} \int_0^4 r^2drd\theta \)

9. A potential \( f \) for the vector field \( \vec{F} = (6x^2 - 5y^2 - 6xz^2, -10xy + 2z^3, -6x^2z + 6yz^2) \) is (i.e. \( \vec{F} = \nabla f \))

A. \( (2x^3 - 5xy^2 - 3x^2z^2, -5xy^2 + 2y^3, -3x^2z^2 + 2yz^3) \)
B. \( 2x^3 - 10xy^2 - 6x^2z^2 + 4yz^3 \)
C. \( (12x - z^2, -10x, -6x^2 + 12yz) \)
D. \( 2x - 6x^3 + 12yz - 6z^2 \)
E. \( 2x^3 - 5xy^2 - 3x^2z^2 + 2yz^3 \)
10. Consider the region in the \(xy\)-plane bounded from above by the curve \(y = 4x - x^2\) and from below by line \(y = x\). The centroid of this region (i.e., center-of-mass of this region for density = 1) is the point

A. \((\frac{54}{5}, \frac{27}{4})\)
B. \((\frac{12}{5}, \frac{3}{2})\)
C. \((2, 4)\)
D. \((\frac{3}{2}, \frac{12}{5})\)
E. \((\frac{27}{4}, \frac{54}{5})\)

11. The volume of the region in the 1st octant bounded by the sphere \(x^2 + y^2 + z^2 = a^2\), the cylinder \(x^2 + y^2 = a^2\) and planes \(z = a, \ x = 0, \ y = 0\) is given by

A. \(\int_0^{\pi/2} \int_0^a \int_0^a 4 \sin \theta \, dz \, dr \, d\theta\)
B. \(\int_0^{\pi/2} \int_0^a \int_0^a r^2 \sin \theta \, dz \, dr \, d\theta\)
C. \(\int_0^{\pi/2} \int_0^a \int_0^a rdz \, dr \, d\theta\)
D. \(\int_0^a \int_0^a \int_0^{a - \sqrt{2-x^2-y^2}} dz \, dx \, dy\)
E. \(\int_0^{\pi/2} \int_0^{\pi/4} \int_0^a \rho^2 \sin \varphi \, d\rho \, d\varphi \, d\theta\)

12. Which of the following statements is correct (assuming that all the functions involved have all orders of derivatives):

A. The divergence of a gradient field is always zero.
B. The divergence of a curl field can be nonzero if the domain is NOT simply connected.
C. The gradient of a divergence (of some vector field) is always a zero vector field.
D. The curl of a gradient field is always a zero vector field.
E. None of the above four statements is correct.
13. Find the maximum of $f = 2x + 7y - 3z$ on the ellipsoid $2x^2 + 7y^2 + 3z^2 = 6$.

14. A parametric surface $S$ is given by $\vec{r} = \vec{r}(u, v)$, $x = u \cos v$, $y = u \sin v$, $z = v$.
   a) Find a unit normal vector $\vec{n}$ of the surface at a general point $\vec{r}(u, v)$.
   [Some intermediate computational steps here could be useful for b) as well.]
   b) Suppose $S$ is defined by $0 \leq u \leq 1$, $0 \leq v \leq 2\pi$. Calculate $I = \int \int_S \sqrt{x^2 + y^2} \, dS$.

15. Consider a solid $E$ in $\mathbb{R}^3$ consisting of all points $(x, y, z)$ satisfying
   \[ z \geq x^2 + y^2 + z^2 - \sqrt{x^2 + y^2 + z^2}. \]
   a) Apply the spherical coordinates $(\rho, \phi, \theta)$ to simplify this messy inequality description of the solid to some simpler expression like $\rho \leq \cdots$.
   b) What are the ranges of the spherical coordinates for the solid?
   c) Based on these results, calculate the volume $V$ of this solid.

16. Let $\vec{F} = (y - x)\hat{i} + (x - z)\hat{j} + (x - y)\hat{k}$ and let $C$ be the boundary of the part of the plane $x + 2y + z = 2$ in the first octant oriented counterclockwise if viewed from point $(1,1,1)$, which is above the plane. Use Stokes' Theorem to evaluate $I = \oint_C \vec{F} \cdot d\vec{r}$.
   [Hint: notice that the plane could be treated as a graph surface.]