Derivations of Useful Trigonometric Identities

**Pythagorean Identity**

This is a basic and very useful relationship which comes directly from the definition of the trigonometric ratios of sine and cosine on the unit circle.

We construct a line segment extending from the origin to the circle in a direction which makes an angle \( \theta \) in the counter-clockwise direction from the positive \( x \)-axis. We then extending a line segment vertically downward from the intersection point on the circle to the \( x \)-axis, thereby forming a right triangle. Since the hypotenuse of this triangle is also a radius of the circle, it has a length of \( 1 \). One leg of the right triangle extends along the \( x \)-axis away from the origin for a distance of \( x = \cos \theta \), the other leg reaches upward from the \( x \)-axis with a length of \( y = \sin \theta \). The intersection point on the unit circle therefore has coordinates \((\cos \theta, \sin \theta)\), which information will be useful to us later.

Because these three sides form a right triangle, we know from the Pythagorean Theorem that \( x^2 + y^2 = 1^2 \). From the description above, we find that we can then also write \((\cos \theta)^2 + (\sin \theta)^2 = 1\). The Theorem we used to obtain this relation given its name to the trigonometric equation, which is referred to as the Pythagorean Identity, \( \sin^2 \theta + \cos^2 \theta = 1 \).

We can divide this identity through by either \( \sin^2 \theta \) or \( \cos^2 \theta \) to produce two alternative forms which are also useful to know:

\[
\frac{\sin^2 \theta}{\cos^2 \theta} + \frac{\cos^2 \theta}{\cos^2 \theta} = \frac{1}{\cos^2 \theta} \quad \Rightarrow \quad \tan^2 \theta + 1 = \sec^2 \theta \quad \text{and} \quad \\
\frac{\sin^2 \theta}{\sin^2 \theta} + \frac{\cos^2 \theta}{\sin^2 \theta} = \frac{1}{\sin^2 \theta} \quad \Rightarrow \quad 1 + \cot^2 \theta = \csc^2 \theta .
\]
**Angle-addition Formulas**

Many trigonometric relationships are based on a key pair of equations referred to as the “angle-addition formulas”. These relations provide the means for calculating the sine or cosine of the sum of two angles, in terms of the sines and cosines of the individual angles.

A useful construction as a starting point for developing these equations is to set up two line segments of length 1, one making an angle \( \alpha \) above the \( x \)-axis, the other at an angle \( \beta \) below that axis. Using the information discussed earlier, we can say that the right endpoint of the upper line segment lies at the coordinates \((\cos \alpha, \sin \alpha)\) and that for the lower segment is at \((\cos \beta, -\sin \beta)\). The distance formula (which is also connected with the Pythagorean Theorem) gives the separation between these two points from

\[
d^2 = (\cos \alpha - \cos \beta)^2 + (\sin \alpha - [-\sin \beta])^2 = (\cos \alpha - \cos \beta)^2 + (\sin \alpha + \sin \beta)^2.
\]

We can now rotate this triangle, without changing its size or shape, so that the lower line segment lies on the \( x \)-axis, placing the right endpoint of that segment now at \((1, 0)\). At the origin, the vertex of the triangle there contains the angle we will represent as \( \Phi = \alpha + \beta \). The opposite endpoint of the upper line segment now has
the coordinates \((\cos \Phi, \sin \Phi)\). The distance between the two endpoints of the line segments in this new situation is given by

\[
d^2 = (\cos \Phi - 1)^2 + (\sin \Phi - 0)^2.
\]

Since this is the same triangle, simply rotated in one diagram relative to the other, this distance is the same given by either equation, permitting us to write

\[
(\cos \alpha - \cos \beta)^2 + (\sin \alpha + \sin \beta)^2 = (\cos \Phi - 1)^2 + (\sin \Phi - 0)^2
\]

\[
\Rightarrow \quad (\cos^2 \alpha - 2 \cos \alpha \cos \beta + \cos^2 \beta) + (\sin^2 \alpha + 2 \sin \alpha \sin \beta + \sin^2 \beta)
\]

\[
= (\cos^2 \Phi - 2 \cos \Phi + 1) + \sin^2 \Phi
\]

\[
\Rightarrow \quad \cos^2 \alpha + \sin^2 \alpha + \cos^2 \beta + \sin^2 \beta - 2 \cos \alpha \cos \beta + 2 \sin \alpha \sin \beta
\]

\[
= \cos^2 \Phi + \sin^2 \Phi - 2 \cos \Phi + 1
\]

\[
\Rightarrow \quad 1 + 1 - 2 \cos \alpha \cos \beta + 2 \sin \alpha \sin \beta = 1 - 2 \cos \Phi + 1
\]

applying the Pythagorean Identity

\[
\Rightarrow \quad -2 \cos \alpha \cos \beta + 2 \sin \alpha \sin \beta = -2 \cos \Phi.
\]

Recalling that \(\Phi = \alpha + \beta\), we can at last write the angle-addition formula for cosine,

\[
\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta.
\]

The construction we have discussed does not lend itself well to finding the corresponding angle-addition formula for sine. Instead we will use the Pythagorean Identity:

\[
\sin^2 \Phi = 1 - \cos^2 \Phi = 1 - (\cos \alpha \cos \beta - \sin \alpha \sin \beta)^2
\]

\[
\Rightarrow \quad 1 - (\cos^2 \alpha \cos^2 \beta - 2 \cos \alpha \cos \beta \sin \alpha \sin \beta + \sin^2 \alpha \sin^2 \beta)
\]

\[
= (\sin^2 \beta + \cos^2 \beta) - \cos^2 \alpha \cos^2 \beta + 2 \cos \alpha \cos \beta \sin \alpha \sin \beta - \sin^2 \alpha \sin^2 \beta
\]

again using Pythagorean Identity; \(\alpha\) could be used instead

\[
= \sin^2 \beta - \sin^2 \alpha \sin^2 \beta + \cos^2 \beta - \cos^2 \alpha \cos^2 \beta + 2 \cos \alpha \cos \beta \sin \alpha \sin \beta
\]

\[
= \sin^2 \beta (1 - \sin^2 \alpha) + \cos^2 \beta (1 - \cos^2 \alpha) + 2 \cos \alpha \cos \beta \sin \alpha \sin \beta
\]

\[
= \sin^2 \beta \cos^2 \alpha + \cos^2 \beta \sin^2 \alpha + 2 \cos \alpha \cos \beta \sin \alpha \sin \beta
\]

by the Pythagorean Identity

\[
= (\sin \beta \cos \alpha + \cos \beta \sin \alpha)^2.
\]
We are nearly finished: we still need to resolve signs, since taking the square root across this last equation leads to \( \sin(\alpha + \beta) = \pm (\sin \beta \cos \alpha + \cos \beta \sin \alpha) \). If we set \( \beta = 0 \), this reduces to
\[
\sin(\alpha + 0) = \sin \alpha = \pm (\sin 0 \cos \alpha + \cos 0 \sin \alpha) = \pm (0 \cdot \cos \alpha + 1 \cdot \sin \alpha) ,
\]
which only yields a correct equation when using the plus sign; we find this is also the case if we set \( \alpha = 0 \). Hence, the angle-addition formula for sine is
\[
\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta .
\]

From an angle-addition formula, we can produce an “angle-difference” formula by adding \( \alpha \) to \(-\beta\), rather than to \(+\beta\). Because the cosine function has even symmetry, while the sine function has odd symmetry, the only effect this has on the original formula is to change the sign of the “\( \sin \) \( \beta \)” factor (since \( \sin(-\beta) = -\sin \beta \)):
\[
\cos(\alpha - \beta) = \cos(\alpha + [-\beta]) = \cos \alpha \cos \beta - \sin \alpha \sin[-\beta] \\
= \cos \alpha \cos \beta - \sin \alpha \cdot (-\sin \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta ;
\]
\[
\sin(\alpha - \beta) = \sin(\alpha + [-\beta]) = \sin \alpha \cos \beta + \cos \alpha \sin[-\beta] \\
= \sin \alpha \cos \beta + \cos \alpha \cdot (-\sin \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta .
\]

**Double-angle Formulas and Consequences**

With the “angle-addition” formulas in hand, we can directly work out relations for multiples of an angle. Those which will be of the most frequent use to us are the formulas for “double-angles”:
\[
\sin(\alpha + \alpha) = \sin(2\alpha) = \sin \alpha \cos \alpha + \cos \alpha \sin \alpha = 2 \sin \alpha \cos \alpha ;
\]
\[
\cos(\alpha + \alpha) = \cos(2\alpha) = \cos \alpha \cos \alpha - \sin \alpha \sin \alpha = \cos^2 \alpha - \sin^2 \alpha .
\]

The “double-angle formula” for cosine can also be written in two alternative forms through the application of the Pythagorean Identity:
\[
\cos(2\alpha) = (1 - \sin^2 \alpha) - \sin^2 \alpha = 1 - 2 \sin^2 \alpha \quad \text{or}
\]
\[
\cos(2\alpha) = \cos^2 \alpha - (1 - \cos^2 \alpha) = 2 \cos^2 \alpha - 1 .
\]

These latter two versions of the double-angle formula for cosine can be re-arranged to yield two relations that will be helpful later on:
\[
\cos(2\alpha) = 1 - 2 \sin^2 \alpha \quad \Rightarrow \quad 2 \sin^2 \alpha = 1 - \cos(2\alpha) \\
\quad \Rightarrow \quad \sin^2 \alpha = \frac{1}{2} \cdot [1 - \cos(2\alpha)] \quad \text{and}
\]
\[
\cos(2\alpha) = 2 \cos^2 \alpha - 1 \quad \Rightarrow \quad 2 \cos^2 \alpha = 1 + \cos(2\alpha) \\
\quad \Rightarrow \quad \cos^2 \alpha = \frac{1}{2} \cdot [1 + \cos(2\alpha)] .
\]
Product-to-sum formulas

It will be of some use on certain occasions to know formulas for the product of the sine or cosine of one angle with the sine or cosine of another. To derive these, we notice that such products appear in the angle-addition formulas, so we can add or subtract the equations in various ways to isolate the product in which we are interested. For example,

\[
\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta \\
+ \cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta
\]

\[
= \cos(\alpha + \beta) + \cos(\alpha - \beta) = 2 \cos \alpha \cos \beta
\]

\[
\Rightarrow \cos \alpha \cos \beta = \frac{1}{2} [\cos(\alpha - \beta) + \cos(\alpha + \beta)] .
\]

The other products are obtained by using the combinations

\[
\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta \\
- \cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta
\]

\[
\Rightarrow \sin \alpha \sin \beta = \frac{1}{2} [\cos(\alpha - \beta) - \cos(\alpha + \beta)] ;
\]

\[
\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta \\
+ \sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta
\]

\[
\Rightarrow \sin \alpha \cos \beta = \frac{1}{2} [\sin(\alpha + \beta) + \sin(\alpha - \beta)] ; \quad \text{and}
\]

\[
\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta \\
- \sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta
\]

\[
\Rightarrow \cos \alpha \sin \beta = \frac{1}{2} [\sin(\alpha + \beta) - \sin(\alpha - \beta)] .
\]

This last formula can also be found by swapping the angles \( \alpha \) and \( \beta \) in the equation for the product \( \sin \alpha \cos \beta \) and using the odd symmetry of the sine function to write \( \sin(\beta - \alpha) \) as \( -\sin(\alpha - \beta) \).

Sum-to-product formulas

While we will not often need them, we should also know about the relations for the sums of sines and cosines of two angles. These require a little work to derive, since we are essentially turning the product-to-sum formulas “inside-out”. (continued)
To accomplish this, we will re-label the angles in those formulas as $\theta$ and $\phi$, so that $\theta + \phi = \alpha$ and $\theta - \phi = \beta$. By solving these little equations simultaneously, we have $\theta = \frac{\alpha + \beta}{2}$ and $\phi = \frac{\alpha - \beta}{2}$. We can now re-write the product-to-sum formulas in this way:

\[
\frac{1}{2} \left[ \sin (\theta + \phi) + \sin (\theta - \phi) \right] = \sin \theta \cos \phi \Rightarrow \sin (\theta + \phi) + \sin (\theta - \phi) = 2 \sin \theta \cos \phi
\]

\[
\Rightarrow \sin \alpha + \sin \beta = 2 \sin \left( \frac{\alpha + \beta}{2} \right) \cos \left( \frac{\alpha - \beta}{2} \right);
\]

\[
\frac{1}{2} \left[ \sin (\theta + \phi) - \sin (\theta - \phi) \right] = \cos \theta \sin \phi \Rightarrow \sin \alpha - \sin \beta = 2 \cos \left( \frac{\alpha + \beta}{2} \right) \sin \left( \frac{\alpha - \beta}{2} \right);
\]

This can also be obtained by replacing $\beta$ with $-\beta$ in the preceding sum-to-product formula and using the odd symmetry of the sine function to write $\sin(-\beta) = -\sin \beta$.

\[
\frac{1}{2} \left[ \cos (\theta - \phi) + \cos (\theta + \phi) \right] = \cos \theta \cos \phi \Rightarrow \cos \alpha + \cos \beta = 2 \cos \left( \frac{\alpha + \beta}{2} \right) \cos \left( \frac{\alpha - \beta}{2} \right);
\]

and (this one requires a bit of extra work)

\[
\frac{1}{2} \left[ \cos (\theta - \phi) - \cos (\theta + \phi) \right] = \sin \theta \sin \phi \Rightarrow \cos (\theta + \phi) - \cos (\theta - \phi) = -2 \sin \theta \sin \phi
\]

switching the order of terms

\[
\Rightarrow \cos \alpha - \cos \beta = -2 \sin \left( \frac{\alpha + \beta}{2} \right) \sin \left( \frac{\alpha - \beta}{2} \right).
\]

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3 – 9 June 2010