Weierstrass’ “Half-Angle” Substitution

Here is a substitution technique that is useful for integrands involving rational functions of sums or differences of trigonometric functions. This “trick” was found by the 19th Century German mathematician Karl Weierstrass, who also made a number of more fundamental contributions to calculus. (If you dealt with the $\varepsilon - \delta$ definition of limits in Calculus I, then you’ve met Karl…) This presentation is based on Problems 57 to 61 from Section 7.4 of Stewart.

Weierstrass noticed that a useful, if obscure, relationship between the sides of a right triangle could be set up if one angle were labeled as a “half-angle” $x/2$ and the legs of the triangle were identified as $t$ and $1$, so that $\tan(x/2) = t/1$. We would then have the other trigonometric ratios

\[
\sin\left(\frac{x}{2}\right) = \frac{t}{\sqrt{1 + t^2}} \quad \text{and} \quad \cos\left(\frac{x}{2}\right) = \frac{1}{\sqrt{1 + t^2}}.
\]

The “double-angle” formulas can then be used to find expressions for trigonometric ratios of the full angle $x$:

\[
\sin x = 2 \sin\left(\frac{x}{2}\right) \cos\left(\frac{x}{2}\right) = 2 \cdot \frac{t}{\sqrt{1 + t^2}} \cdot \frac{1}{\sqrt{1 + t^2}} = \frac{2t}{1 + t^2},
\]

\[
\cos x = 1 - 2 \sin^2\left(\frac{x}{2}\right) = 1 - 2 \cdot \left(\frac{t}{\sqrt{1 + t^2}}\right)^2 = 1 - \frac{2t^2}{1 + t^2} = \frac{1 + t^2 - 2t^2}{1 + t^2} = \frac{1 - t^2}{1 + t^2}.
\]

Lastly, the defining relationship $\tan\left(\frac{x}{2}\right) = t \Rightarrow x = 2 \tan^{-1} t$ can be differentiated to give us

\[
\frac{dx}{dt} = \frac{d}{dt} (2 \tan^{-1} t) = 2 \cdot \frac{1}{1 + t^2} \quad \Rightarrow \quad dx = \frac{2}{1 + t^2} \, dt.
\]

This idea gives us a set of substitutions that allow us to turn integrals of rational functions with sums or differences of trigonometric functions into integrals of rational functions of polynomials, integrals we have a better understanding of how to solve. We will follow this up with some examples.
58) \( \int \frac{dx}{3 - 5 \sin x} \rightarrow \int \frac{\frac{2}{1 + t^2} \ dt}{3 - 5 \left( \frac{2t}{1 + t^2} \right)} \cdot \frac{1 + t^2}{1 + t^2} = \int \frac{2}{3(1 + t^2) - 5 \cdot 2t} \ dt \\
= \int \frac{2}{3t^2 - 10t + 3} \ dt = \int \frac{2}{(3t - 1) \cdot (t - 3)} \ dt = 2 \cdot \int \frac{-\frac{3}{4}}{3t - 1} + \frac{\frac{1}{4}}{t - 3} \ dt \\
= -\frac{3}{4} \ln|3t - 1| + \frac{1}{4} \ln|t - 3| + C \rightarrow \frac{1}{4} \ln \left| \frac{\tan(\frac{x}{2}) - 3}{3 \tan(\frac{x}{2}) - 1} \right| + C .

59) \int \frac{1}{3 \sin x - 4 \cos x} \ dx : \ \text{solution similar to that of \#58}

60) \int_{\pi/3}^{\pi/2} \frac{1}{1 + \sin x - \cos x} \ dx \quad \text{x:} \quad \pi/3, \quad \pi/2 \\
t = \tan(x/2) : \quad \tan(\pi/6) = 1/\sqrt{3}, \quad \tan(\pi/4) = 1 \\
= \int_{1/\sqrt{3}}^{1} \frac{1}{\frac{2t}{1 + t^2} - \frac{1 - t^2}{1 + t^2}} \ dt = \int_{1/\sqrt{3}}^{1} \frac{2}{(1 + t^2) + 2t - (1 - t^2)} \ dt \\
= \int_{1/\sqrt{3}}^{1} \frac{2}{2t^2 + 2t} \ dt = \int_{1/\sqrt{3}}^{1/2} \frac{1}{t^2 + t} \ dt = \int_{1/\sqrt{3}}^{1/2} \frac{1}{t} - \frac{1}{t + 1} \ dt \\
by \text{method of partial fractions} \\
= \ln|t| - \ln|t + 1|\bigg|_{1/\sqrt{3}}^{1} = \ln\left|\frac{t}{t + 1}\right|_{1/\sqrt{3}}^{1} = (\ln \frac{1}{2} - (\ln \frac{1/\sqrt{3}}{1 + [1/\sqrt{3}]}) \\
= \ln \left( \frac{1 + [1/\sqrt{3}]}{2 \cdot [1/\sqrt{3}]} \right) = \ln \left( \frac{\sqrt{3} + 1}{2} \right) .
\[ 61) \int_0^{\pi/2} \frac{\sin 2x}{2 + \cos x} \, dx = \int_0^{\pi/2} \frac{2 \sin x \cos x}{2 + \cos x} \, dx \]

\[ x: \quad 0 \to \pi/2 \quad t = \tan(x/2): \quad \tan 0 = 0 \quad \tan(\pi/4) = 1 \]

\[ = \int_0^1 \frac{2 \cdot \frac{2t}{1 + t^2} \frac{1 - t^2}{1 + t^2}}{2 + \frac{1 - t^2}{1 + t^2}} \cdot \frac{2}{1 + t^2} \, dt \]

\[ = 8 \cdot \int_0^1 \frac{t \cdot (1 - t^2)}{(1 + t^2)^2 \cdot [2(1 + t^2) + (1 - t^2)]} \, dt = 8 \cdot \int_0^1 \frac{t \cdot (1 - t^2)}{(1 + t^2)^2 \cdot (t^2 + 3)} \, dt \]

While we could apply the method of partial fractions at this point, it might be easier to work with this integrand if we first make the substitution \( u = t^2 \Rightarrow du = 2t \, dt \):

\[ t: \quad 0 \to 1 \quad u = t^2: \quad 0 \to 1 \]

\[ \to 8 \cdot \int_0^1 \frac{(1 - u) \cdot \frac{1}{2} \, du}{(1 + u)^2 \cdot (u + 3)} = 4 \cdot \int_0^1 \frac{1 - u}{(1 + u)^2 \cdot (u + 3)} \, du \]

\[ = 4 \cdot \int_0^1 \frac{-1}{1 + u} + \frac{1}{(1 + u)^2} + \frac{1}{u + 3} \, du = 4 \left[ -\ln|u + 1| - \frac{1}{u + 1} + \ln|u + 3| \right]_0^1 \]

\[ = 4 \left[ (\ln \left| \frac{1 + 3}{1 + 1} \right| - \frac{1}{1 + 1}) - (\ln \left| \frac{0 + 3}{0 + 1} \right| - \frac{1}{0 + 1}) \right] = 4 \cdot (\ln 2 - \frac{1}{2} - \ln 3 + 1) \]

\[ = 4 \cdot \left( \frac{1}{2} + \ln \frac{2}{3} \right) = 2 + 4 \ln \frac{2}{3} \quad \text{or} \quad 2 + \ln \frac{16}{81} \]

-- G. Ruffa

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